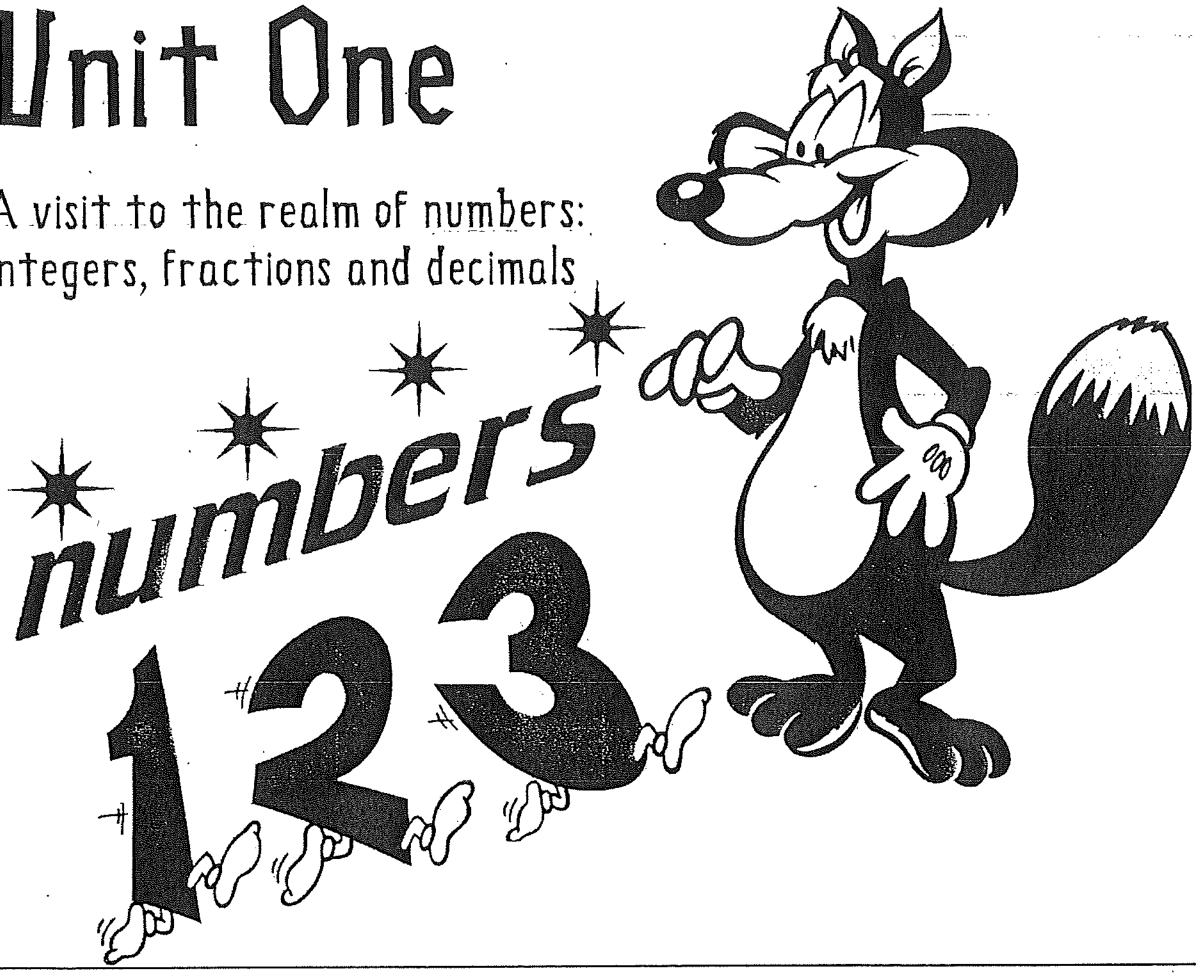
Unit One

A visit to the realm of numbers: integers, fractions and decimals



1.1. Basic notions ★

Whole numbers are called **integers.**

Numbers can be **even** : 2, 4, 6, 8, 10, 12 and so on.
Or **odd**. 3, 5, 7, 9, 11, 13 and so on.

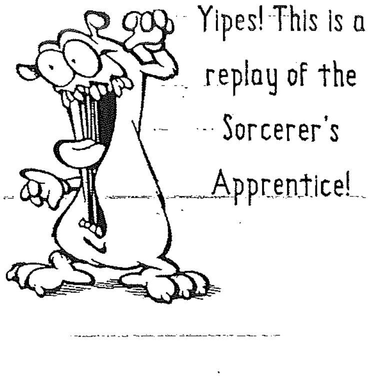
We can do many things with numbers, and that is the subject of this book. Here we look at simple operations.

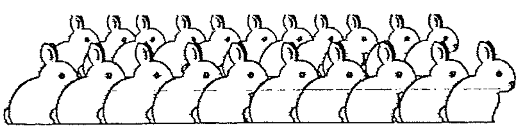
We can **multiply** them: 2 x 2 = 4.

Multiplication is shown by the letter x, or by using a dot: •

Hence:

3 x 6 = 18 or 3 • 6 = 18
(Three times six equals/is equal to eighteen.)
(Three multiplied by six equals/is equal to eighteen.)



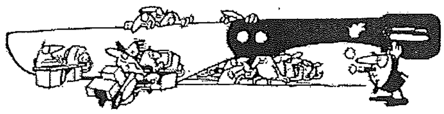


This is also **multiplication**.

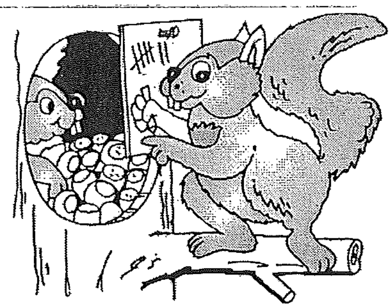
We can **divide** them: $4 \div 2 = 2$
Or: $2/2 = 1$

This is **division.**

We show division by using a fraction bar (/) or the symbol $\div$.

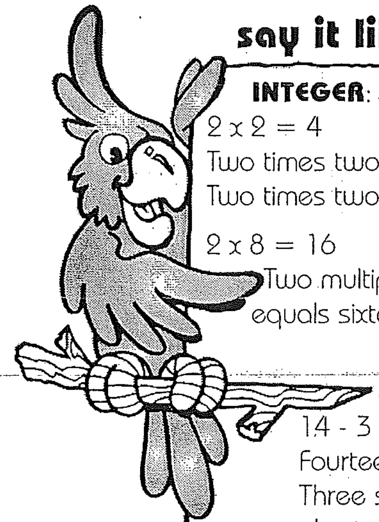


Cutting the staff in half



We can **add** numbers:
$2 + 2 = 4$
$3 + 11 = 14$
This is **addition.**

And we can **subtract** them:
$2 - 1 = 1$
$14 - 3 = 11.$
This is **subtraction.**
We subtract 3 from 14 to get 11.



say it like this

INTEGER: IN-te-jer

$2 \times 2 = 4$
Two times two equals four.
Two times two is four.

$2 \times 8 = 16$
Two multiplied by 8 equals sixteen.

$2 + 2 = 4$
Two plus two equals four.
Two added to two equals four.

$4 - 2 = 2$
Four minus two equals two.
Two subtracted from four is/equals/is equal to two.

$14 - 3 = 11$
Fourteen minus three is/equals/is equal to eleven.
Three subtracted from fourteen is/equals/is equal to eleven.
Three subtracted from fourteen yields eleven.

THE LANGUAGE OF MATHEMATICS

to multiply	to multiply one number by another [multiplication]
to divide	to divide one number by another [division]
to add	to add two numbers, several numbers; to add one number to another [addition]
to subtract	to subtract one number from another: subtract 3 from 9 to get 6. [subtraction]
hence	therefore, thus: $1 + x = 10$, hence (therefore) $x = 9$

1.2. Working through the operations ★

The description of math operations, or the way we are taught to reason them through, varies from one country to the next. Here is just one way we do math operations in English. It is not the only way, but it will give you an idea of how we learned to do them as children in the United States.

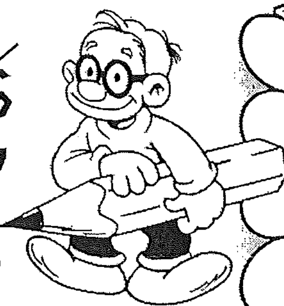
A. Addition

$$\begin{array}{r} {}^{(1)} \\ 26 \\ 34 \\ 27 \\ \hline 87 \end{array}$$

Six plus four is ten plus seven is seventeen. Bring down the seven and carry the one (to the tens column). One plus two is three plus three is six plus two is eight. Bring down the eight. The sum of the three numbers is thus eighty-seven.

B. Subtraction

$$\begin{array}{r} {}^{(3)} \\ 4\cancel{6} \\ 27 \\ \hline 19 \end{array}$$



It is not possible to take seven from six (well actually, it is, but the result is a negative number which we shall not get involved with here). So I borrow ten from the tens column. Now I have sixteen. Sixteen minus seven is nine. Bring down the nine. Three minus two is one. Bring down the one. The remainder (or difference) is nineteen.

C. Multiplication

$$\begin{array}{r} {}^{(1)}26 \\ {}^{(4)}\times 82 \\ \hline 52 \\ {}^{(1)}208 \\ \hline 2132 \end{array}$$

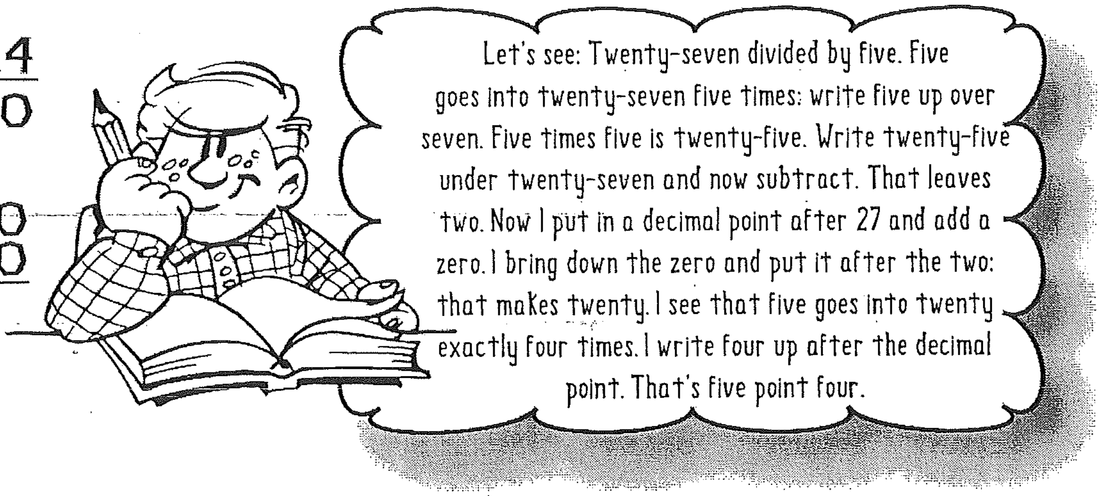
I start with the last digit of eighty-two. Two times six is twelve: bring down the two and carry the one. Two time two is four plus one: that makes five.

Now the first digit of 82: eight times six is forty-eight: bring down the eight under the five and carry the four. Eight times two is sixteen plus four, making twenty. Now I add: bring down the two. Add five and eight to get thirteen: bring down the three and carry the one. One and zero, that's one. Now bring down the two. That's two thousand one hundred and thirty-two.

Division follows on the next page.

D. Division

```
   5.4
5 | 27.0
    25
     20
     20
```



1.3. Factors ★★

1.4. If the integer m divides a evenly, then a is divisible by m, or m is a **factor** of a.

If m is a factor of a, there is another integer, n, such that

$a = m \times n$.

12 = 4 x 3 ; 12 is a **multiple** of 4 and also of 3.

Say this

12/1 = 12
12/2 = 6
12/3 = 4
12/4 = 3
12/6 = 2

Complete:

1, 2, 3, 4, and 6 are

Factor this! $2a^3 - 2a^2b + 2ab$

result:

$2a(a^2 - ab + b^2)$

common monomial factor (or monomial divisor)

Factor this! $6a^2b^2 - 12a^3b + 30a^3b^4$

- greatest common divisor (G.C.D.) of 6, 12, and 30 is 6.
- G.C.D. of a^2, a^3, and a^3 is a^2.
- G.C.D. of b^2, b, b^4 is b.

∴ largest monomial factor is: $6a^2b$

result: $6a2b(b - 2a + 5ab^3)$

Exercise 1.

Talk about factors and multiples for the following integers.

1. 10
2. 13
3. 100

Exercise 2.

Factor the following, explaining the steps.

1. $2a^2 - 6a^4$
2. $15x^3 - 10x^2 + 5x$
3. $5m^2 - 10m$

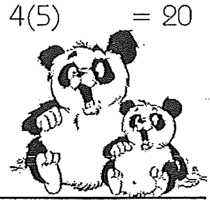
1.4. Rules for multiplication and division ★★

In addition and multiplication, the order doesn't matter! That's neat!

5+7= 12
7+5=12
4x(2x3)=24
(4x2)x3=24

That's because addition and multiplication are **commutative** *and* **associative** *(the grouping does not matter). And* **distributional** *with respect to addition:*

4(3+2) = 12 + 8 = 20
= 4(5) = 20



Say these operations out loud.

1 + 6 = 7	6 + 1 = 7
2 + 3 = 5	3 + 2 = 5
1 + 2 + 3 = 6	3 + 2 + 1 = 6
2 x 6 = 12	6 x 2 = 12
2 x 6 x 2 = 24	6 x 2 x 2 = 24
2 •(6 x 2) = 24	(2 x 6) • 2 = 24

Think it out! Say it out loud! ★★

Answer the following question, explain your reasoning out loud.

1. Above you see that for multiplication and division the "order doesn't matter." Qualify this statement. HINT: consider 2(4+5) + 11, which normally gives 29. But I performed it (erroneously) in such a way as to get 24. Explain. Why do we use parentheses?

Exercise 3. ★ and ★★

To answer the questions you will need to use all the language and terms used in the preceding sections.

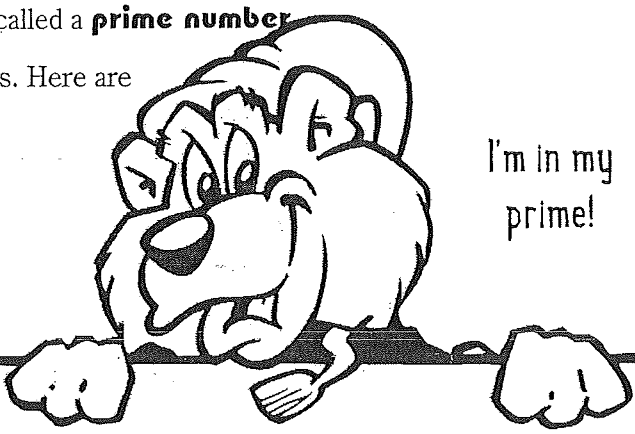
1. A number of operations will give us the final sum of 2. For example: 2 + 0 = 2. And also: 4 - 2 = 2; 100 ÷ 50 = 2; or 1,000 − 998 = 2. Describe at least 5 operations that will give the final sum of 4. (★)
2. What can you say about these integers: 1, 3, 7 ? (★)
3. What can you say about these integers: 2, 4, 8 ? (★)
4. What can you say about this operation: 12 ÷ 11 as opposed to 12 ÷ 6? (★)
5. Why is 2 + 3 + 4 the same as 3 + 2 + 4? (★★)
6. 2(3+6) = 18: (2x3) +(2x6) = 18; or 2x9=18. Explain why this is so. (★★)

1.5. Prime numbers ★

Any whole number (integer) is divisible by itself and by 1. If p is a whole number greater than 1, which has only p and 1 as factors, then p is called a **prime number**.

2, 3, 5, 7, 11, 13, 17, 19 and 23 are all primes. Here are some others:

29	43	61	79
31	47	67	83
37	53	71	89
41	59	73	97



QUESTIONS

Show and explain the reasons for your answers:

1. Are only odd numbers prime numbers?
2. Are all odd numbers prime numbers?
3. Why is 14 not a prime number?
4. True or false: The prime numbers cannot be divided evenly by any integers other than themselves and 1.

Think and talk exercise. Think it out! Say it out loud!

Answer the following question, explain your reasoning out loud.

Is there a formula or procedure for finding prime numbers? The following has been suggested. Let n equal any counting number (thus, any positive whole number).

$n \cdot (n+1) + 17$ will give a prime number.

Explain this procedure verbally, that is: what you do to obtain the prime number.

Will it yield all prime numbers? (HINT: make a list from 1 through 15).

1.6. More exciting facts about numbers. ★★★

Numbers can also be:

positive: $+2,\ x,\ 4ab^2$

negative: $-2,\ -1/2y$

rational and irrational! (see pages 23-24 and 26)

and imaginary: $\sqrt{-1}, \sqrt{-7x^2}$

Say it like this.

$\sqrt{-1}$

the square root of minus one

Be an eager beaver! Hurry, turn the page for more exciting, thought-provoking facts about numbers. You'll see that they can really be complex!

Numbers can also be **COMPLEX**.

A **complex number** is a number having the form $a + bi$, where a and b are real numbers. Examples of complex numbers:

$2 - 4i$

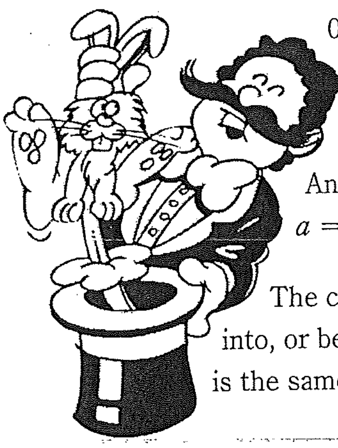
$\sqrt{5} + \sqrt{3 - 4}$

$-6 + \sqrt{-7}$

Wait a minute! According to the rule just stated the second term must be in the form bi. In the last example we have only the i, it seems!

Can you answer the question? Hint: b is a coefficient of i.

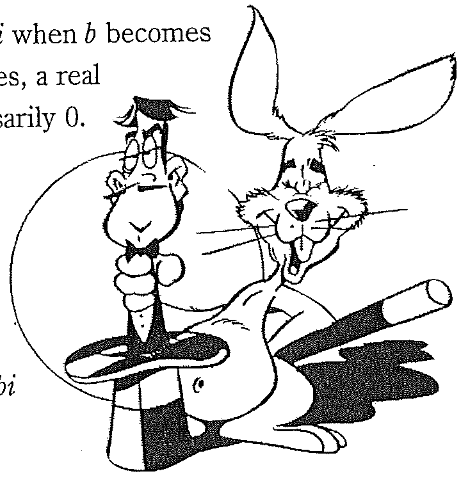
Presto! It's not magic; it's math.



Consider this question: what happens to $a + bi$ when b becomes 0? It is transformed into, or becomes, a real number! (Because 0 x i is necessarily 0. So the bi term disappears...)

And what happens to $a + bi$ when $a = 0$?

The complex number is transformed into, or becomes, imaginary! (Because $0 + bi$ is the same thing as bi.)



Operations involving complex numbers:

Add:
$4 + 3i$ and $-6 - i$

$$\begin{array}{r} 4 + 3i \\ \underline{-6 - i} \\ -2 + 2i \end{array}$$

Multiply:
Add: $3 - 4i$ by $2 + 3i$

$$\begin{array}{l} 3 - 4i \\ \underline{2 + 3i} \\ 6 + 8i \\ \underline{\quad + 9i - 12i^2} \\ 6 + i - 12i^2 \end{array}$$

Divide:
Add: $4 - 2i$ by $1 + 3i$

Writing the quotient in fraction form: $\dfrac{4 - 2i}{1 + 3i}$
Multiplying both numerator and denominator by $1 - 3i$:

$$\frac{4 - 2i}{1 + 3i} \times \frac{1 - 3i}{1 - 3i} = \frac{-2 - 14i}{1^2 - (3i)^2} = \frac{-2 - 14i}{10}$$

$$= \frac{-1 - 7i}{5}$$

These binomials are called **conjugate complex numbers.**

1.7. Going a little farther:

Note how we say it.

7x : "seven times x"

7(x+3)
"Seven times the sum of x plus 3"

Now you say it.

a/ 4x – 2y	d/ 4x – 2y
b/ 6(2x-3)	e/ 6(2x-3)
c/ 3(x+2y)	f/ 3(x+2y)

Also note the terms we use in talking about addition and subtraction.

the addends → 2, +8
the sum → 10

the minuend → 8
the subtrahend → –2
the remainder → 6

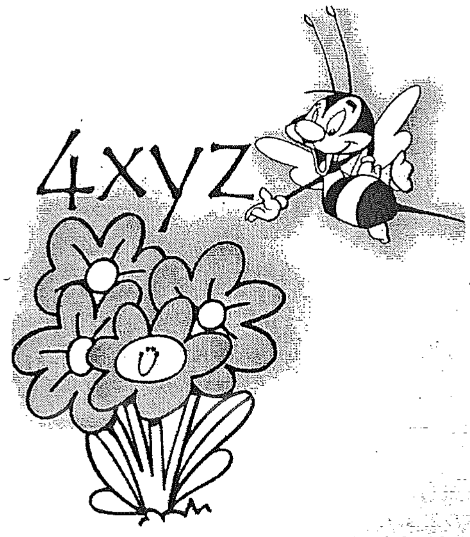
Also note the terms we use in talking about multiplication and division.

the multiplicand → 2
the multiplier → x 8
the product → 16

the dividend → 8
the divisor → 2
the quotient → = 4

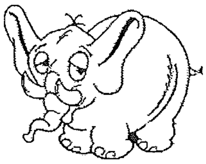
1.8. Going a lot farther:

Take the product 4xyz.
4 is the numerical **coefficient** of xyz.
4 xy is the coefficient of z.
4yz is the coefficient of x.



Types of expressions:

monomial: x, 2x, $2x^2y$ (one single term)
binomial: 2x-6; -y+2x; $2-y^2$ (two terms)
trinomial: $x^2 + y^2 - 6$; x3 - 4 + 2x (three terms)
polynomial: 2 or more terms



Exercise 4. Don't forget! Review and Recall Opportunity

Are the following three statement correct? If not, please correct them.

1. The multiplicand is the number that you multiply by to get your quotient.

2. The dividend is the number you need to divide by to get your product.

3. The subtractor is the number you subtract, and the result is the remainder.

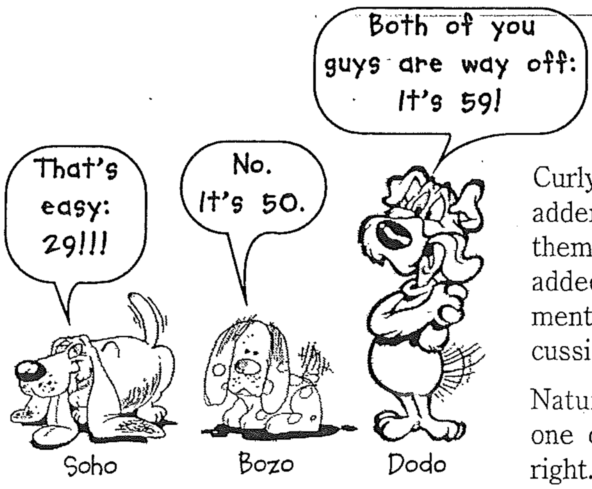
Now explain these mistakes:

4. $7(4 + 2) = 30$

5. $6 + 4(3 - 2) = 16$

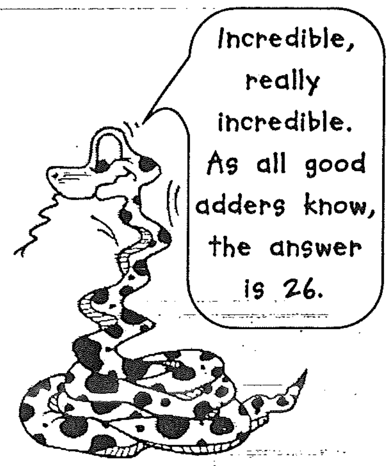
Creature feature

One day we asked three friends of ours to find the answer to this problem: $6 + 4(6 - 1) = ?$



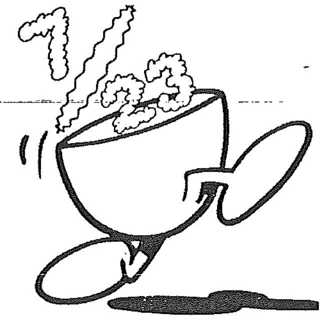
Curly, the friendly adder, overheard them and quickly added his comments to the discussion.

Naturally, only one of them was right.



Explain why 3 of our friends are wrong and how they were able to make the mistakes!

1.9. Fractions and Decimals ★



Say it like this.

$\frac{1}{2}$	one-half	$\frac{6}{7}$	six-sevenths
$\frac{1}{3}$	one-third	$\frac{7}{10}$	seven-tenths
$\frac{1}{4}$	one-fourth	$\frac{5}{11}$	five-elevenths
$\frac{2}{3}$	two-thirds	$\frac{7}{23}$	seven-twenty-thirds
$\frac{3}{4}$	three-fourths		

Note also these ways of saying fractions:

2/3 = two-thirds, or two over three

3/22 = three twenty-seconds, or three over twenty-two

x/y = x over y; x divided by y.

$\frac{x+1}{y-6}$ = x plus one over y minus six.

Now you say these

1.	$^{4}/_{5}$	6.	$^{67}/_{103}$
2.	$^{8}/_{9}$	7.	$^{31}/_{32}$
3.	$^{9}/_{11}$	8.	$^{1}/_{2000}$
4.	$^{11}/_{12}$	9.	$^{1}/_{8,000,000}$
5.	$^{34}/_{35}$	10.	$^{1}/_{10,000,000,000}$

Exploring the realm of fractions

a. basic terms ★

the numerator $^{3}/_{8}$ the denominator

$^{3}/_{0}$ Division by zero is **undefined**, or **not defined**. Thus this fraction is also undefined.

All fractions have a numerator and a denominator, also called the **terms** of the fraction!

b. basic principles ★

Note the following principles, noting the language used, then fill in the blanks in the statement which follows.

$^{3}/_{1}$

A fraction with 1 as the denominator is the same as, or is identical to, the whole number which is its numerator.

$^{12}/_{3} = 4$

Any whole number, a, is represented by a fraction with a numerator equal to a times the denominator.

Now you should be able to fill in the blanks.

$^{3}/_{3}$

If the numerator is ______________ the denominator, the fraction ______________ one. (*Answer at the bottom of the page.*)

c. Reciprocals ★

One fraction is a reciprocal of another if their product is 1. The reciprocal of 2 is thus 1/2. Therefore:

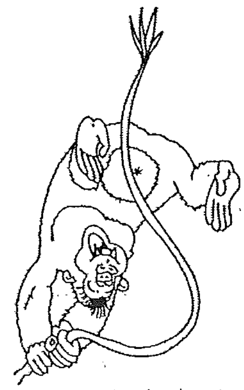
$^{1}/_{2} \bullet {}^{2}/_{1} = 1$

What's the reciprocal of 4/21?

What fraction can you multiply 4/21 by to get 1?

$^{4}/_{21} \bullet {}^{21}/_{4} = 1$

To find the reciprocal of any fraction, interchange the numerator and denominator, or invert the fraction.



I am inverted. Am I therefore a reciprocal?

Consequently, in the expression

$^{7}/_{21} \div {}^{18}/_{31}$

7/21 is the ______________ and

18/31 is the ______________.

(Answers at the bottom of the page)

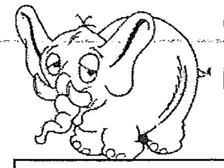
Now: to divide the dividend by the divisor, simply invert the divisor and multiply:

$^{7}/_{21} \div {}^{18}/_{31}$

$^{7}/_{21} \bullet {}^{31}/_{18} = {}^{217}/_{378}$ **(= .5741)**

ANSWERS

If the numerator is the same as the denominator, the fraction is equal to (or: equals) 1. The answers to the second question: 7/21 is the *dividend* and 18/31 is the *divisor*.



Exercise 5. Don't forget! Review and Recall Opportunity

Note the operation: $21 \div 7 = 3$.

1. 21 is the ________________.
2. 7 is the ________________.
3. 3 is the ________________.
4. The numerator and denominator are also called the __________ of a fraction.

Note the operation: $a \bullet b = c$.

5. a is the ________________.
6. b is the ________________.
7. c is the ________________.
8. $a \bullet b$ ____________ $b \bullet a$. This is true because ________________________ ________________.

If you multiply the numerator and denominator (the terms) of a fraction by the same non-zero number, the value of the fraction **remains the same** (or: **remains unchanged**).

$$2/3 \bullet 6/6 = 12/18$$

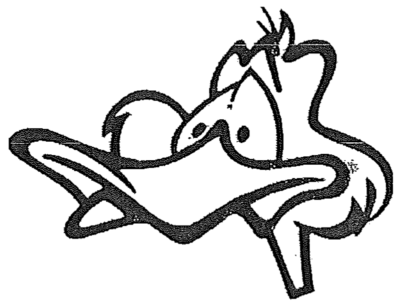
$$a/x \bullet y/y = ay/xy$$

If you divide the numerator and denominator of any fraction by the same non-zero number, the value of the fraction **remains the same**.

$$12/30 \div 6/6 = 2/5$$

$$2/3 \div 3/3 = \frac{2/3}{1}$$

Why would anybody want to go to all this trouble?



Explain and justify this:

$$\frac{a}{x} \div \frac{y}{y} = \frac{a}{x} \bullet \frac{y}{y} = \frac{ay}{xy}.$$

What happens to the y's in the terms? They **cancel out**.

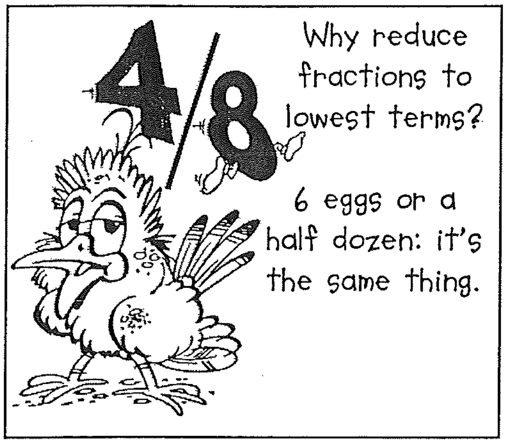
$$\frac{1\cancel{y}}{2\cancel{y}}$$

THE LANGUAGE OF MATHEMATICS

any	any whole number, any integer, any fraction
to be equal to	3/1 is equal to 3.
to be the same as	3/1 is the same as 3.
to invert	Invert 2/3 to obtain 3/2 (the reciprocal).
to interchange	When we interchange the numerator and denominator, 2/3 becomes 3/2.
to remain unchanged	When we multiply any integer by 1, the value remains unchanged (it remains the same).
to cancel (out)	The y's cancel out. We indicate this using **cancellation marks**.

d. lowest terms ★★

A fraction is in lowest terms when its numerator and denominator have no common integral factor except 1. A fraction not in lowest terms may be reduced to lowest terms by dividing the numerator and denominator by their common factors.



Quick check: Are these in lowest terms? Why or why not? (*See the bottom of the page.*)

$1/2$ $\quad$ $3/4$ $\quad$ $3/6$

$3/8$ $\quad$ $9/21$

To reduce a fraction to lowest terms cancel all the common factors of the numerator and the denominator.

Similarly, to add fractions, all fractions must have a common denominator.

$1/7 + 2/3$ $\quad$ What does the value 21 represent in adding these two fractions?

Think and Talk Exercise: Think it out! Say it out loud! ★★

Are the following true or false? Justify your answers.

1. A fraction is in lowest terms when its numerator and denominator have no common integral factor except 1.
2. We can reduce a fraction to lowest terms by dividing the numerator and denominator by any factor of either term.
3. To divide one fraction by another, it is first necessary to change both to equal fractions having a common denominator.
4. To add or subtract fractions it is first necessary to convert them to equal fractions having a common denominator.

Read and understand ★★★

Read the 3 definitions below and then identify the expressions or equations labeled A-E below.

1. A mixed expression is one with a fraction and one or more that are not fractions.
2. A complex fraction is one that contains a fraction in the numerator or denominator, or both.
3. A fractional equation is one in which some function of the unknown appears in the denominator.

A $\dfrac{1+\dfrac{s}{l}}{1-\dfrac{s^2}{l^2}}$

B $\dfrac{x}{x+4}+\dfrac{1}{x}=3$

C $3x-5-\dfrac{1}{3x}=0$

D $2x-4-\dfrac{1}{3}$

E $3x-5-\dfrac{1}{3x}$

ANSWER:
3/6 and 9/21 are not in lowest terms because 3 is a common factor in the numerators and denominators of both.

e. decimals ★

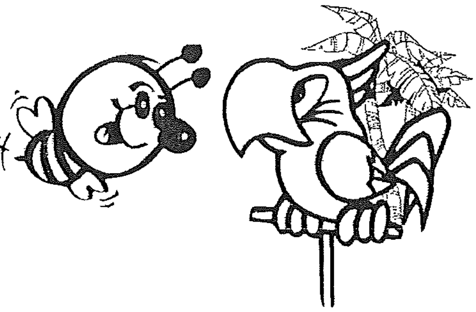
I don't like them. They get in my way. Decimal points, I mean. Frankly, I don't see their utility. But that's only my opinion.

. =	a decimal point
1.2 =	one point two
.51 =	point five one, or point fifty-one.

The digits following a decimal point are called a decimal fraction.

.501
.5624 } Every decimal fraction represents a fraction.
.32

You get a decimal fraction by dividing the numerator by the denominator.



Okay! Now let me ask you one: How do you find the fraction which a decimal fraction represents?

Think and Talk Exercise: Read it through; think it out; Say it out loud!

1. Take the fraction whose decimal is 10 and whose numerator is the first digit to the right of the decimal point.

2. Take the fraction whose denominator is 100 and whose numerator is the second digit to the right of the decimal point.

3. Take the fraction whose denominator is 1,000 and whose numerator is the third digit to the right of the decimal point.

Continue this procedure until you have used each digit to the right of the decimal point. The denominator in each step is 10 times the denominator in the previous step!

.5624

This decimal fraction is shown **to the fourth decimal place.**

About rounding numbers

.257 can be **rounded out** or **off** to .26.
We can **round up** or **round down**: 2.44 can be rounded down to 2.4, 2.9 can be rounded up to 3.
Numbers can also be rounded to the nearest tenth, hundreth, thousandth, and so on.

1.10. Algebraic equations ★★

To solve the problem posed in the story which follows you need the basic notions of algebra. However, some of the information on the preceding pages can be used to justify the operations involved. Before you look at the solution below, try to reason it through, and say out loud how you would solve the problem.

Fenwick was a mathematical genius, and many say that this explains how one day he won $200 million. Once the tax office applied a substantial subtrahend and the rules of subtraction, however, Fenwick found that his remainder was $80 million.

Now Fenwick had two sons, Bick and Bock. His love for Bick was > that for Bock. Bock's temper was 10> than Bick's. So Fenwick was very careful in wording his will: the sum of $80 million was to be divided into two parts such that the sum of 1/6 of the smaller and 1/7 of the larger was to be 12. The larger share was to go to Bick.

The next day while he was on his way home, he was killed by a falling meteor (a subsequent unit will deal with the probability of such an event...)

The attorney is trying to decide how much will go to the brothers who are getting impatient.

The solution:

Divide $80 million into two parts or shares.

Let x = the smaller part.

Let $80 - x$ = the larger part.

Write an equation.

$$\frac{1}{6}x + \frac{1}{7}(80 - x) = 12$$

$$7x + 6(80 - x) = 504$$

$$7x + 480 - 6x = 504$$

$$x = 24$$

Bick gets the larger share: $80 - $24 = $56 million and Bock gets $24 million. Bock didn't believe that this was the correct solution. But Bick proved it.

$\frac{1}{6}24 = 4;$ $\frac{1}{7}56 = 8$ $4 + 8 = 12$

A new problem for you:

Say all the steps out loud as you work out this simple algebra problem. (You should also write out your solution).

1/2 some number = 1/3 that same number +17. Find that number. Don't forget the proof!

1.11. Common symbols encountered in mathematical descriptions ★★★

Now is a good time to see whether you know what these symbols describe. Match Column A (the symbol) with its equivalent expression in Column B. (*Answers at the back of the book.*)

column A

1.	$\pm$	......	15.	∞	......
2.	$=$	......	16.	$\sum$	......
3.	$\neq$	......	17.	$\int$	......
4.	$>$	......	18.	$\int_b^a$	......
5.	$\gg$	......	19.	$\cong$	......
6.	$<$	......	20.	$\cup$	......
7.	$\ll$	......	21.	$\cap$	......
8.	$\geq$	......	22.	$\subset$	......
9.	$\leq$	......	23.	$\supset$	......
10.	$\not>$	......	24.	$\in$	......
11.	$\not<$	......	25.	$\notin$	......
12.	$\approx$	......	26.	$\varnothing$	......
13.	$\equiv$	......	27.	!	......
14.	$\sim$	......			

column B

a. is identical to
b. is much greater than
c. is much less than
d. is equal to, equals
e. is not equal to
f. plus or minus
g. is greater than
h. is less than
i. the union of 2 sets
j. is approximately equal to
k. is similar to
l. the sum of
m. integral, the integral of
n. is congruent to
o. is less than or equal to
p. is greater than or equal to
q. the intersection of 2 sets
r. is included in, is a subset of
s. null (empty) set
t.. is not an element of
u. is an element of
v. contains as a subset
w. factorial
x. the integral taken between the values a and b of the variables
y. is not greater than
z. is not less than
aa infinity

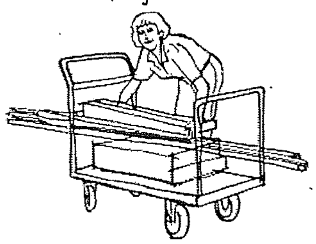
Other common symbols encountered:

$\propto$	is proportional to
$x_1 x_2$	subscript: x subscript 1, or x one
$x'\,y'$	prime: x prime, y prime
Δx	"delta x" : the change in x
$\bar{\ }$	upper bar to indicate "average": $\bar{v}$, $\bar{a}$ (average velocity, average acceleration)

Exercise 6.

Say the following out loud, or explain whether the statements are true or false.

I thought that the length ≤ the cart, but the cart ≪ than they are.



1. $6+2 < 7\text{-}5$ ______
2. $5/2 \neq 11/3$ ______
3. $\pi \approx 3.14$ ______
4. $x > 5$ ______
5. $x \geq y+2$ ______
6. $x \equiv y$ ______
7. $4^2 > 2^3$ ______

More difficult notations:

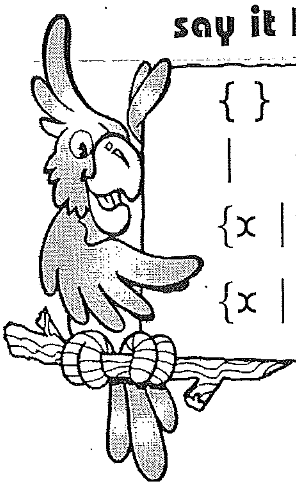
8. $\mathrm{E} = 0$ for $r > R_0$: $\mathrm{E} \neq 0$ for $r < R_i$.

9. $\{\mathbf{F}, \mathbf{F}'\} \equiv \{\mathbf{F}+\mathbf{F}\}$ (note: **F** represents a vector)

10. $\sum_{\text{seq}} = {}^{n}\!/\!_{2}(a+l)$ (sum of the numbers of an arithmetic progression (sequence), where $n =$ the number of terms or members, a = the first term, l = the last term of the series.

1.12. Common expressions dealing with sets ★★★

The following expressions may be commonly found in mathematical proofs or definitions.



say it like this

{ } braces. Read: *the set of all ...*

| vertical bar. Read: *such that*

$\{x \mid x > 1\}$ The set of all x such that x is greater than 1.

$\{x \mid x \leq -1\}$ The set of all x such that x is less than or equal to -1.

1. $\{x\}$
2. $\{x \mid x > 5\}$
3. $\{x \mid x \neq y\}$
4. $\{x \mid x \leq 10\}$

Unit 1: Final exercises and activities ★

1. Think of all operations that will yield 4, and not exceeding 2 digits! You should be able to find at least 3 operations involving addition, subtraction, multiplication and division. For instance:

   3+1
   5−1
   4 x 1
   4 ÷ 1

2. The following is a well known number trick. The directions are given on the left and shown mathematically on the right.

[1]	Choose any whole number.	n
[2]	Add 5.	$n+5$
[3]	Multiply by 2.	$2(n+5) = 2n + 10$
[4]	Subtract 4.	$2n + 6$
[5]	Divide by 2.	$n + 3$
[6]	Subtract the initial number	$n+3-n$
[7]	The result will always be 3;	3.

Now read out these two different number tricks that are shown algebraically.

[1]	n	[1]	n
[2]	$n+3$	[2]	$3 \times n = 3n$
[3]	$2(n+3) = 2n+6$	[3]	$3n + (n+1) = 4n + 1$
[4]	$2n + 6 + 4 \; (= 2n + 10)$	[4]	$4n + 1 + 11 = 4n + 12$
[5]	$\frac{2n + 10}{2} = n + 5$	[5]	$\frac{4n + 12}{4} = n + 3$
[6]	$n+5 - n = 5$	[6]	$n+3 - 3 = n$

THE LANGUAGE OF MATHEMATICS
Summary of all important language

Numbers

integer
whole number
odd
even
digit
factor
common monomial factor
largest monomial factor
a multiple of
prime number
imaginary number
complex number
conjugate complex number
negative/positive number

Other useful terms

coefficient
monomial
binomial
trinomial
polynomial

Operations

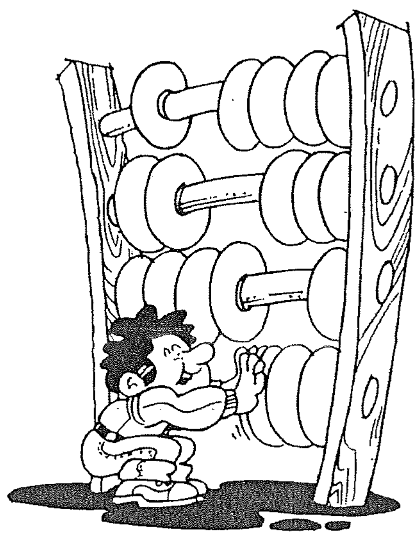
to add: add x to y, add x and y.
addition
addends
the sum
to subtract: subtract x from y.
subtraction
minuend
subtrahend
the differnce = the remainder
to multiply: multiply y by x
multiplication
multiplicand
multiplier
the product
to divide: divide y by x
division
dividend
divisor
greatest common divisor
the quotient
division by zero is undefined

Fractions

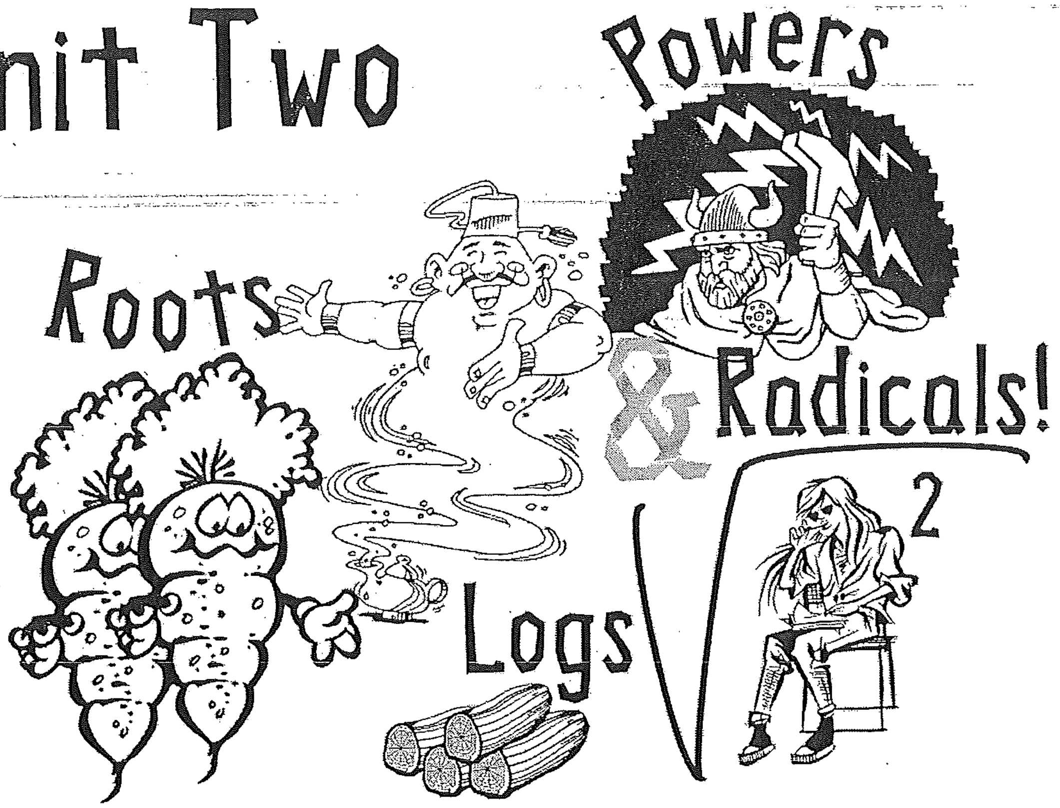
decimals
decimal point
decimal fraction
numerator
denominator
the terms of a fraction
reciprocal
to interchange
to invert

Consequently

therefore
thus
hence



Unit Two



2.1. Exploring the realm of powers and exponents ★

You are probably familiar with the following. Can you say them? Cover the expressions on the right and try to say each term.

a^2	*a* squared, or: the square of *a*
a^3	*a* cubed, or: the cube of *a*
a^4	*a* (raised) to the fourth power

The subject of powers is unquestionably difficult. These are the elementary power laws. The real tough stuff is coming.

To generalize a bit: ★

b is any number
n is a whole number greater than 0.
b^n means the product of n factors each of which is equal to b.

b^n b raised to the nth power
b^2 b squared
b^3 b cubed

a^n ← exponent; a ↑ base

$a^0 = 1$ (a to the power zero is 1, if $a \neq 0$)
$a^{-1} = 1/a$ (a to the power minus 1 is 1 over a)
$a^{-2} = 1/a^2$ (a to the power minus 2 is 1 over a squared)

Quick check: Exercise 1. ★★

Explain the following. Say it out loud and write it down.

① $a^{-n} = \dfrac{1}{a^n}$

② $\left(\dfrac{x}{y}\right)^{-1} = \dfrac{y}{x}$

Exploring the power laws a bit further. ★★

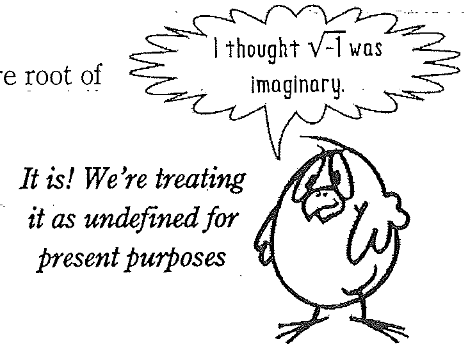
$a^m \bullet a^n = a^{m+n}$	By the addition law of exponents: add exponents to multiply a power by another power.
$a^m/a^n = a^{m-n}$	By the subtraction law of exponents, subtract exponents to divide a power by another power.
$(a^m)^n = a^{mn}$	**To raise a power to a power**, multiply the powers. This is the multiplication law of exponents. We raise a power to another power by multiplying the exponents.
$(ab)^m = a^m b^m$	To raise a product to a power, raise each factor to the power.
$(a/b)^m = a^m/b^m$	To raise a fraction to a power raise the numerator and denominator to the power.

Exercise 2. ★★

Read the following out loud. You should then write out your explanation.

1. $b^n \bullet b^m = b^{n+m}$
2. $(ab)^n = a^n \bullet b^n$
3. $(a^m)^n = ?$
4. $x^{-2} + y^0$
5. $(ab)^{-3} + x^3 \text{-} y^0$
6. $4^a - b^m$

2.2. Exploring the realm of roots



We take the square root of any nonnegative number, the square root of a negative number being undefined for practical purposes.

$\sqrt[n]{x}$ = the principle nth root of x

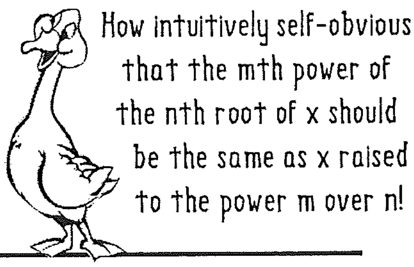
Note these basic relationships:

$a^{1/2} = \sqrt{a}$: a to the power 1/2 is equivalent to the principal square root of a.

$a^{1/3} = \sqrt[3]{a}$: a to the power 1/3 is equivalent to the principal cubic root of a.

$$x^{m/n} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$$

The statement $x^{m/n}$ means the mth power of the nth root, or the nth root of the mth power.



$$x^{-m/n} = \frac{1}{x^{m/n}}$$

A negative fractional exponent is equal to the reciprocal of the number to the positive exponent.

Decimal powers, fractional exponents and roots ★★

$$a^{.4} = a^{4/10} = a^{2/5}$$

$$a^{.7} = a^{7/10}$$

Decimal powers should be changed into their equivalent fractional exponents.

$$a^{2/5} = \sqrt[5]{a^2}$$

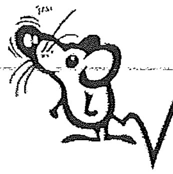
power (numerator) — root (denominator)

The numerator of a fractional exponent denotes the power of the base. The denominator denotes the root of the base.

Exercise 3. ★★

Say these out loud.

1. $x^{7/9}$
2. $a^{3/5}$
3. $b^{x/y}$
4. $\sqrt{9}$
5. $\sqrt{28^5}$
6. $\sqrt{81}$
7. $\sqrt{x^3y^9}$
8. $\sqrt{(a-b)^3}$
9. $\sqrt{(x-y)+c}$
10. $a^{-x/y}$
11. $m^{-ab} + x^{3/4} - y^{-2/3}$
12. $x^{-1/3}$



2.3. Going a little farther ★★

A radical is an indicated root of a number. Hence, $\sqrt{25}$, $\sqrt{9}$, $\sqrt{4}$, and $\sqrt{m}$ are radicals. The numbers 25, 9, 4, and m are radicands. $\sqrt[n]{\ }$ is the radical sign. The index, or order, of the radical is the figure or letter written above the radical sign indicating which root is to be taken.

I agree that I am a radical, but I am not an indicated root of a number. And I don't care to be simplified.

Simplifying radicals: ★★★

Some radicals can be simplified by finding their indicated roots; others can be simplified by changing them into other radicals. Factor the radicand into two factors, one being the greatest perfect power of the same degree as the radical. Take the indicated root of this factor that now becomes a coefficient of the resulting surd (see below).

$\sqrt{20} = \sqrt{4}\cdot\sqrt{5} = 2\sqrt{5}$

$\sqrt[3]{24x^4} = \sqrt[3]{8x3} \cdot \sqrt[3]{3x}$

$= 2x\sqrt[3]{3x}$

The radicand does not contain a fraction and the index of the radical is the smallest possible number.

Exercise 4.

TRUE OR FALSE? Are the following statements true or false. Explain.

1. $\sqrt{xy} = \sqrt{x} + \sqrt{y}$ ______
2. $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ ______
3. $4^{1/2} = 2$ ______
4. $(a + b)(a + b) = a^2 + b^2$ ______

2.4. Going a lot farther: exploring the realm of rational vs. irrational numbers ★★

Isn't that an uncanny coincidence! Just like people numbers can be rational or irrational. I'm getting all wound up in this!

$\sqrt{4} = 2$ is a **rational number**. A rational number can be expressed in the fraction form a/b in which a is a positive or negative integer and b is a positive integer. $\sqrt{2}$ is an **irrational number**. An irrational number that is the indicated root of a rational number is a **surd**.

Is that where they get the word ABSURD from????

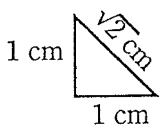
Exercise 5: Recall and review opportunity ★★

Complete the following sentences.

1. Any number expressed in fraction form is ______________________
2. To get 8 from 2, using only powers, we ______________________
3. To get 16 from 2 using only powers we ______________________
4. A number containing one or more variables in irreducible radical form is ______________
5. The root of a given number is ______________________
6. The sign √ is called ______________________
7. A number under this sign is called ______________________
8. The number or other indication of the root to be taken is called ______________
9. A sum containing one or more irrational roots of numbers is ______________

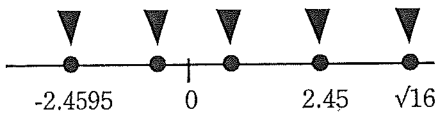
There is an infinite number of irrational numbers. Contrary to rational numbers that can be expressed as a fraction and that can be a point on the scale of numbers irrational numbers have no exact position on such a scale. Irrational numbers are nevertheless very real.

According to the Pythagorean theorem, the hypotenuse of the triangle shown here is an irrational number.



Another example of an irrational number is π ("pi" : it rhymes with "eye").

The infinite scale of real numbers



Question:

If every rational number corresponds to a point on the infinite scale of numbers, does every point on this scale correspond to a rational number?

2.5. Scientific notation ★

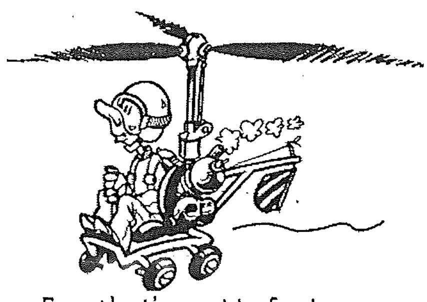
Notice how we read out quantities written in scientific notation.
The speed of light is:

2.997929×10^{10} cm sec^{-1}

This can be **rounded out** to:

3×10^{10} cm sec^{-1}

Three times ten to the tenth power per second.



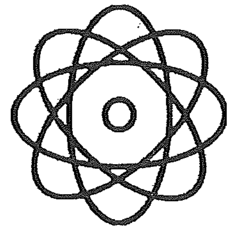
Gee, that's pretty fast.

Other examples:

3.6×10^{-6}	three point six times ten to the negative sixth power
4.1×10^{20}	four point one times ten (raised) to the twentieth power

Say these out loud

1. Planck's constant: $= 6.6252 \times 10^{27}$ ergs sec $^{-1}$
   $= 6.6262 \times 10^{34}$ J.S
2. One parsec: $= 1.92 \times 10^{12}$ miles
   $= 3.08 \times 10^{13}$ km
   $=$ 206,265 AU (astronomical units)
3. One light year $= 5.88 \times 10^{12}$ miles
   $= 9.46 \times 10^{12}$ km
4. The electron electron mass $= 9.110 \times 10^{-31}$ kg
   charge $= -1.60219 \times 10^{-19}$ coulomb
5. The proton: mass $= 1.6724 \times 10^{-27}$ kg
   charge $= 1.60219 \times 10^{-19}$ coulomb



2.6. Exploring the realm of logarithms ("logs") ★

Logarithms are exponents as the following presentation reveals.

3 is the exponent of the base 10.

$$\mathbf{10^3 = 1000}$$

10 is the base.

The logarithm of 1000 to the base 10 is 3.

Note the following:

$2^3 = 8$	The exponent of 2 is 3. The logarithm of 8 to the base 2 is 3.
$3^4 = 81$	The base is 3 and the logarithm is 4. $\text{Log}_3\ 81 = 4$ (the logarithm of 81 to the base 3 is 4.)
$2^5 = 32$	The base is 2 and the logarithm is 5. $\text{Log}_2\ 32 = 5$ (the logarithm of 32 to the base 2 is 5).

Now say these out loud

1. $\log_{10} 1000 = 3$
2. $\log_{10} 10{,}000 = 4$
3. $\log_2 8 = 3$
4. $\log_7 49 = 2$
5. $\log_{16} 256 = 2$
6. $\log_4 64 = 3$
7. $\log_9 81 = 2$
8. $\log_3 81 = 4$

Unit 2 End Games: a formal mathematical proof. ★★★

Proof that √2 is not a rational number.

To prove: √2 cannot be expressed as the quotient of two integers.

1. Either √2 is rational or it is not rational.
2. Given: a rational number can be expressed as a fraction.
   Suppose that $\sqrt{2} = a/b$ where a/b is reduced to lowest terms.
3. If $\sqrt{2} = a/b$, $2 = a^2/b^2$, and $a^2 = 2b^2$.
4. Since 2 is a factor of $2b^2$, it is a factor of a^2.
5. If 2 is a factor of a^2, it is a factor of a.
6. If 2 is a factor of a, it is twice a factor of $2b^2$.
7. If 2 is twice a factor of a^2, it is twice a factor of $2b^2$.
8. If 2 is twice a factor of $2b2$, it is a factor of b^2.
9. If 2 is a factor of b^2, it is a factor of b.
10. From (5) and (9), 2 is a factor of a and b, and a/b is not in lowest terms, which is contrary to hypothesis.
11. From (1) and (10), √2 is not rational.

Unit 2: Final review and recall exercises.

Do you remember everything? Now is the time to check.

Exercise 1. Say the following out loud. ★

1. x^7
2. 10^{-14}
3. 3×10^{27}
4. 4×10^{10}
5. 1.6×10^{-13}
6. 2.74×10^{-5}
7. 6.42×10^{17}
8. 2.01×10^{13}
9. $x^2 = y^2 = 2 - xy^7$
10. $(a^3 + b) = x^2y^2$

Exercise 2. Complete the following statements. ★

1. In $\sqrt[3]{8}$: 3 is ____________, 8 is ____________, and ± 2 is ____________.
2. In $10^4 = 10{,}000$: the log(arithm) of 10,000 ____________.
3. Explain: $x^{1/2} = \sqrt{x}$ ____________.
4. Explain: $x^{m/n} = (\sqrt[n]{x})^m = \sqrt[n]{x^m}$ ____________.

Exercise 3. What can you say about the following? ★★

1. $\sqrt{125}$ vs. $\sqrt[3]{125}$

2. $\sqrt{-4}$ vs. $\sqrt[3]{-8}$

Exercise 4. Explain the steps in simplifying the radical ★★★

1. $5\sqrt{72x} = 5\sqrt{36} \cdot \sqrt{2x}$
   $5\sqrt{36} \cdot \sqrt{2x} = 30\sqrt{2x}$

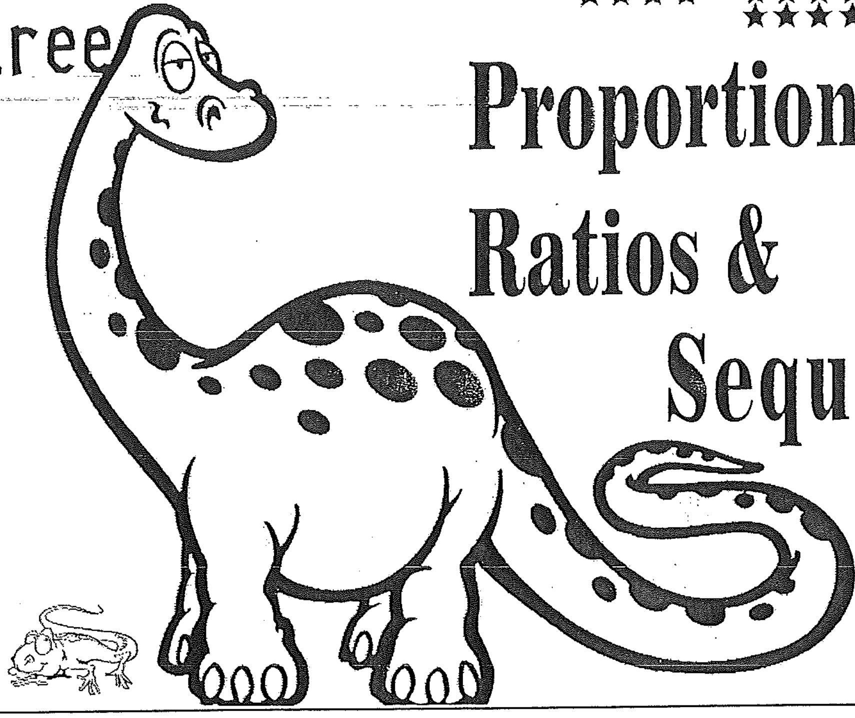
2. $\sqrt{\frac{1}{8}} = \sqrt{\frac{2}{16}} = \frac{\sqrt{2}}{\sqrt{9}} = \frac{\sqrt{2}}{4}$

THE LANGUAGE OF MATHEMATICS
Summary of all important language

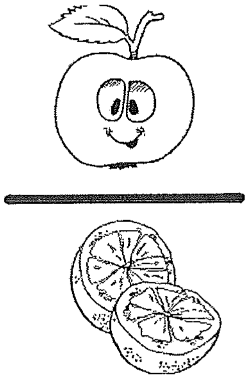
square
cube
power
to raise to the mth power
exponent
base
square root
the principal nth root of x
radical
radicand
radical sign
to simplify a radical
index
order
rational number
irrational number
surd
scale, scale of numbers
scientific notation
hypotenuse of a triangle
logarithm (log)

Unit Three

Proportions, Ratios & Sequences



3.1. Ratios and proportions ★★



A **ratio** is the numerical value of one quantity divided by the numerical value of a second **like** quantity. This means that you cannot relate apples to oranges.

Ratios are indicated as fractions 7/10 or using a colon: 7:10. We say, *the ratio of 7 to 10.*

A **proportion** is an equation, the 2 members of which are ratios.

$\frac{a}{b} = \frac{c}{d}$ *Formerly written as:* a : b :: c : d *"a is to b as c is to d"*

a and *d* are **the extremes** *b* and *c* are **the means**

Quick check! Can you say this? See unit 1, and then, and only then, the bottom of the page.

$\frac{a}{b} = \frac{c}{d}$ $ad = bc$

Solution:
The product of the means is equal to the product of the extremes. Finding the products ad *and* bc *is also called* cross-multiplying the proportion.

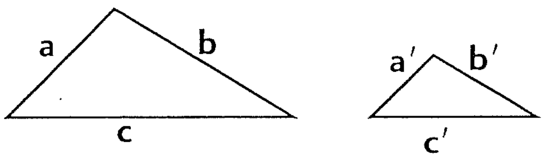
Ratios and proportions: stated differently

1. A ratio is a comparison of two numbers by division. A proportion is a statement that two ratios are equal.
2. Any two units of measurement of the same quantity are directly proportional.
3. Proportions may also be written: **a** $\propto$ **F** (a is proportional to **F**: acceleration is proportional to the force)

3.2. Going a little farther ★★★

$$\frac{a}{b} = \frac{b}{c}$$

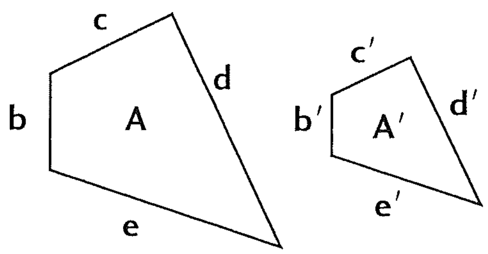
b is the **mean proportional**; it is used for the second and third terms of a proportion.
c is the **third proportional** to *a* and *b*.



Reminder: a′ is read: "*a prime.*"
The corresponding sides of similar polygons are proportional: *a* is **proportional to** *a*′ because triangle abc is **similar to** triangle a′b′c′

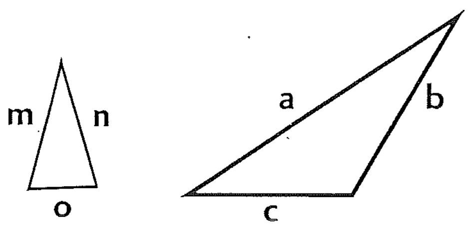
Exercise 1. ★

Comment on the following figures, expressions or equations.



1. The sides of the two polygons shown:
$$\frac{c}{c'} = \frac{d}{d'} = \frac{e}{e'} = \frac{b}{b'}$$

2. The areas (A and A′) of the two polygons shown:
$$\frac{A}{A'} = \frac{c^2}{c'^2} = \frac{d^2}{d'^2} = \frac{e^2}{e'^2} = \frac{b^2}{b'^2}$$



3. Triangle *mon* is not .... triangle *acb*. ...., sides *m, n* and *o* .... sides *a,b* and *c*.

$$\frac{A}{B} = \frac{C}{D}$$

4. We know that $ad = bc$. Say this equation out loud and explain why this is true.

3.3. Variations ★★

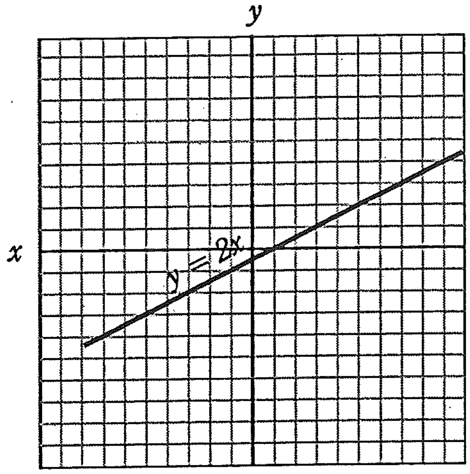
A given value (variable) changes as another value (variable) changes. Note these two examples:

$y = 2x$

y	−4	−2	0	+2	+4
x	−8	−4	0	+4	+8

$y = -2x$

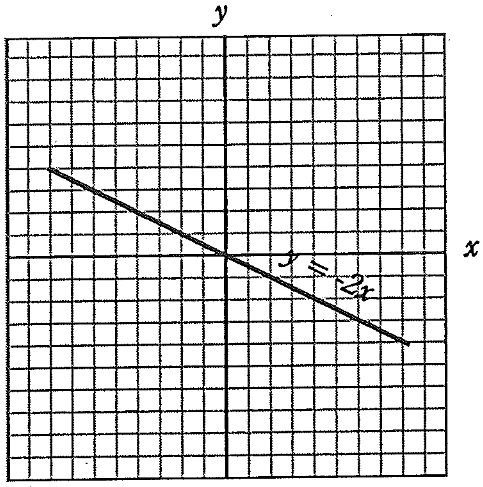
y	−4	−2	0	+2	+4
x	+8	+4	0	−4	−8



An equation can be shown **graphically**.

We **graph** an equation using given values for x and y.

For more work see Unit 5 Functions and Graphs.



A. Direct variation

Direct variation can be expressed by a **linear equation**, whose graph is a straight line. For the equation $y = 2x$, the line has a **slope** of 3. y **varies directly** as twice the value of x.

Direct variation can also be expressed algebraically as $y = kx$, with k being **a constant**. Y increases as x increases if k is positive; y decreases as x increases if k is negative.

B. Inverse variation

The two values change but their product **remains constant**:

xy = c
y = c/x
x = c/y
$y = \pi/r^2$: y varies directly **as** π and inversely **as** the square of r.
$xy = k$: x and y vary **inversely** as k, a constant.

Note: The graph of $xy = k$, a second-degree equation, is an equilateral hyperbola. (*Geometric aspects are not treated here. See* The Language of Geometry).

C. Joint variation

$A = xy$: A varies jointly **as** x and y.

$A = 1/2\ bh$ A varies jointly **as** one-half the product of b and h.

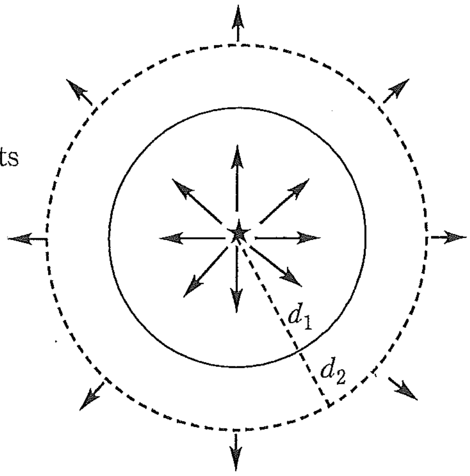
$A = Mv^2$ A varies jointly **as** M and the square of v.

Note the use of AS here and above!

One variable varies jointly **as** two or more variables when it varies **as** the product of these variables.

The inverse square law ★★:

Light energy radiates away from the source (H). Energy passing through a unit area decreases as the square of the distance from the source. At distance $d1$ and $d2$, the amounts of energy received, $l1$ and $l2$ are in the proportion:



$$\frac{l_1}{l_2} = \frac{4\pi d_1^2}{4\pi d_2^2} = \left(\frac{d_1}{d_2}\right)^2$$

Exercise 2. Answer these questions. ★★

1. In the case of the inverse square law above, we see that the amount of energy (E) at a given point is $E/4\pi d^2$. Express this variation.
2. In the above equations, 4π is eliminated from the last terms on the right. Why?
3. Express the following using appropriate language from the preceding page:
   $y = 1/2x$
   Y ________________ as x ________________. x and y are the two ____________________, and 1/2 is ____
4. Express the following in the same way:
   $y = -1/2x$
   Now y ________________ as x ________________ because ________________
5. State the variation in x and y in the following:
   $xy = -7$
   ____
   $2y = x$
   ____

New write out the following ★★ :

1. V varies inversely as P ________________
2. d varies directly as the square of t. ________________
3. S varies directly as x to the fourth power and inversely as the square root of y. ________________
4. M varies jointly as the square of p and the square root of v. ________________
5. The ratio of x_1 to x_2 is proportional to the quotient of m divided by n. ________________
6. The heat capacity (C) of an object varies directly as the amount of heat which must be applied (ΔQ) and inversely with the temperature change (ΔT). ________________
7. The square of P is in direct proportion to the cubes of a. ________________
8. F varies directly as the product of m and v^2, and indirectly as r. ________________
9. A varies jointly as one-third the quotient of the square of x and the square of y. ________________

3.4. Progressions (or sequences) ★

A. ARITHMETIC PROGRESSIONS

An **arithmetic progression** (or sequence): 6, 9, 12, 15, 18, ...
We can determine the following terms by adding 3, the **common difference**.

Question:

How can you find the last term, or any given term, say the 20th of this sequence?

Let a denote the first term.

Let l denote the last term.

Let d denote the common difference.

The first term is	a
The second term is	$a + d$
The third term is	$a + 2d$
The fourth term is	$a + 3d$
The fifth term is	$a + 4d$
The nth term is	$a + (n - 1)d$

$$\boxed{l = a + (n - 1)d}$$

What can you say about the coefficient of d in any given term?

Read and understand ★

The arithmetic mean
The arithmetic mean is the term between any two nonconsecutive terms of an arithmetic progression. Now before continuing, look at the examples on the right and do the brief exercise. Now think: here are 3 terms of a progression: 3, 6, 9, ... How does the arithmetic mean relate to the two nonconsecutive numbers?

Use that information to do this:

1, 3, 5, 7, 9, ...

2, 4, 6, 8, 10, ...

Give 2 examples of arithmetic means in each of the sequences

B. GEOMETRIC PROGRESSIONS ★

A **geometric progression**: 2, 4, 8, 16, 32, ...
Any one of the successive terms can be determined by multiplying the preceding term by the same number, the **common ratio**.

Can you answer these?

1. In the progression above, each successive term is obtained by ____________________

____

2. How does 4 relate to 2? ____________
   How does 8 relate to 2? ____________
   How does 16 relate to 2? ____________
   How does 32 relate to 2? ____________

Question:

How can we determine the last or any subsequent term of this progression? It is up to you to reason this out. Continue in the next column.

Now describe what each letter designates:

____ a ____________________.

____ r ____________________.

____ l ____________________.

The first term is	________
The second term is	________
The third term is	________
The fourth term is	________
The fifth term is	________
The nth term is	________

$$\boxed{l =}$$

What observation will you make that is similar to your observation in the preceding section?

The geometric mean ★★

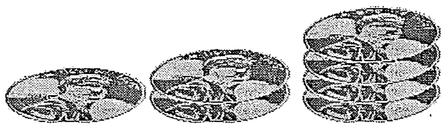
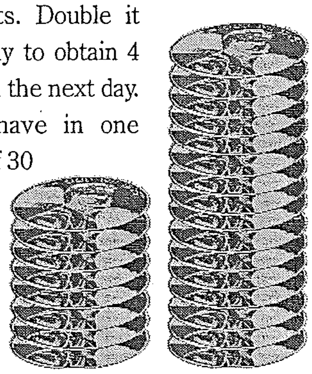
The geometric mean of two numbers is the one and only term between two nonconsecutive terms of a geometric progression. The geometric mean of two numbers is the **mean proportional** between them. Hence, the geometric mean of 1 and 25 is ± 5, and that of 5 and 125 is ± 25.

The geometric mean of two terms is thus the length of the side of a square of area equal to the product of the two numbers. For N numbers it is defined as

$$\sqrt[N]{x_1 x_2 x_3 x_4 \ldots x_N}$$

C. The binary sequence: one type of geometric progression ★

Start with one penny (=$.01). Double the sum the next day. You now have 2 pennies, or 2 cents. Double it again the following day to obtain 4 cents. Double the sum the next day. How much will you have in one month's time? (Assume a month of 30 days.)

DAY 1 DAY 2 DAY 3 DAY 4 DAY 5

Applying the formula for geometric progressions:

$$l = ar^{n-1}$$

we find that on the 30th day you will have amassed 536,870,912 cents, which is equivalent to $5,368,709.12. In a binary sequence each successive term is double the preceding term.

Another important characteristic of the binary sequence ★

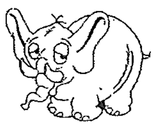
counting number	binary sequence				
1	1	2	4	8	...
2	1	2	4	8	...
3	1	2	4	8	...
4	1	2	4	8	...
5	1	2	4	8	...
6	1	2	4	8	...
7	1	2	4	8	...
8	1	2	4	8	...
9	1	2	4	8	...
10	1	2	4	8	...

Look at the table on the left carefully. On the left are the counting numbers from one to ten. On the right, the first four terms of the binary sequence is shown opposite each number, with one of its terms circled. What correlation or pattern do you detect? Try to formulate this pattern in clear language, then look in the answer section at the back of the book.

Question:

Why is this pattern of special importance to us? (*Again, look in the answer section!*)

D. The sequence of squares ★



Quick recall: (see the answer section)

① 1, 3, 5, 17, 19, 21, 101 are examples of ________ numbers; 2, 4, 6, 18, 22, 66, 132 are examples of ________ numbers.

② 2, 3, 4, 5, 6, 7, 8, 9 are ____________ numbers; 2, 4, 7, 9, 11, 14 are ____________ numbers.

③ $25 = 5^2$: 25 is ________________ of 5; 5 is the ________________ of 25.

Step 1. Note the following and fill in the blanks with the appropriate words. (*See the answer section.*)

$$1 = 1 = 1^2$$
$$1 + 3 = 4 = 2^2$$
$$1 + 3 + 5 = 9 = 3^2$$
$$1 + 3 + 5 + 7 = 16 = 4^2$$
$$1 + 3 + 5 + 7 + 9 = 25 = 5^2$$
$$1 + 3 + 5 + 7 + 9 + 11 = 36 = 6^2$$

The sums of consecutive odd numbers beginning with 1 are ________. When the consecutive odd numbers are ________, their sum is a ________, whose ________ is equal to the number of ________ in the original sequence of odd numbers. From the pattern above, we can predict that $1+3+5+7+9+11+13+15+17 = \_\_\_\_^2$.

Step 2. What is the pattern? Consider the counting numbers from 1-10 and their squares.

number	square	new number
1	1	1
2	4	4
3	9	9
4	16	7
5	25	7
6	36	9
7	49	4
8	64	1
9	81	9
10	100	1

A new number can be obtained from the square of the consecutive positive whole numbers. How? This new number has a special name. What is it?

To find it add the single digits of the square, adding resulting single digits so as to have one single digit. Follow this procedure for the number 8.

$8^2 = 64$: ________________ =

What other striking pattern emerges when the digital roots from 1-8 are examined?

E. Read and understand: The Fibonacci sequence ★★

Read the description of how this sequence is formed, and write out the operation in the space provided on the right.

The first two terms of this sequence must be one. Each succeeding term is the sum of the last term and the preceding term. The 10th term will thus be 55. What do you observe about every third term?

3.5. The binomial theorem ★★★

This theorem provides a way of expanding any integral power of a binomial. Performing the multiplication indicated by the integral powers of each binomial gives:

$$(x+y)^0 = 1$$
$$(x+y)^1 = x + y$$
$$(x+y)^2 = x^2 + 2xy + y^2$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^4$$
$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$$

Let n represent the exponent of $(x + y)$.
The first term is x^n and the last term is y^n.
The second term is $nx^{n-1}y$.
In each successive term the exponent of x **decreases by 1** while the exponent of y **increases by 1**.
If in any term the coefficient is multiplied by the exponent of x, and divided by the number of that term, the result is the coefficient of the next term.
The number of terms is $n + 1$.

Pascal's Triangle ★★

Two sides are made up of 1's. Numbers in each row give the coefficients of the expansion. The first and last number in each row is 1. To determine each of the other numbers in a given row, add each of the 2 numbers nearest it in the row above.

See the End Games for more interesting work relating to Pascal's triangle.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Question:
How does this triangle relate to the binomial expansion shown above and on the next page?

The binomial formula

From the preceding discussionwe may derive this formula:

$$(a+b)^n = a^n + \frac{na^{n-1}b}{1} + \frac{n(n-1)a^{n-2}b^2}{2!} + \frac{n(n-1)(n-2)a^{n-3}b^3}{3!} + \frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{4!} + \ldots + b^n$$

An example:

Expand $(x+y)^4$ by the binomial formula

$$(x+y)^4 = x^4 + \frac{4x^3y}{1} + \frac{12x^2y^2}{2!} + \frac{24xy^3}{3!} + \frac{24y^4}{4!}$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

Finding any term of an expansion. Look at the expansion on the left and fill in the boxes below.

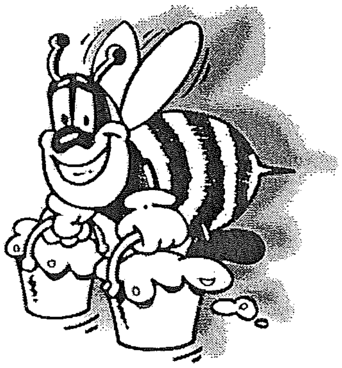
Write the 2nd term.

Write the 3rd term.

Write the 4th term.

Write the 5th term.

It's been a very hard day. 7! of us, visiting 12! flowers and ending up with only 6! grams of honey.



factorial seven

7!

1 x 2 x 3 x 4 x 5 x 6 x 7
= 5,040

Think and Talk exercise. Think it out! Say it out loud! ★★

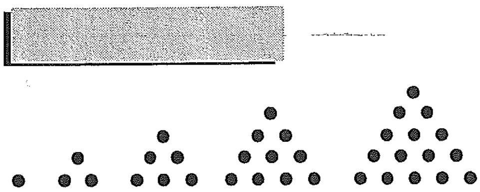
Now examine the four terms you have written. What can you say about:

1. the exponent of y and the number of the term?
2. the exponent of x and the exponent of y.
3. the number of factors in both the numerator and denominator of the coefficient with respect to the exponent of y for a given term?

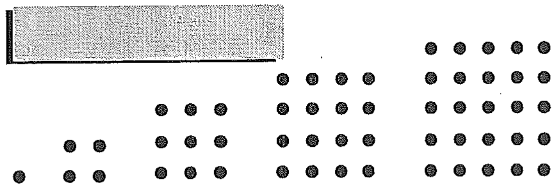
Unit 3 End Games: sequences ★

Answer the questions.

The pattern on the right should be familiar to you now: the sequence of squares. But what about the sequence suggested by the triangles on the left? Write the first 7 terms of each sequence.



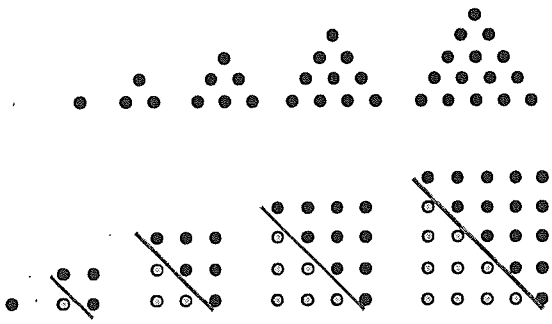
Sequence 1

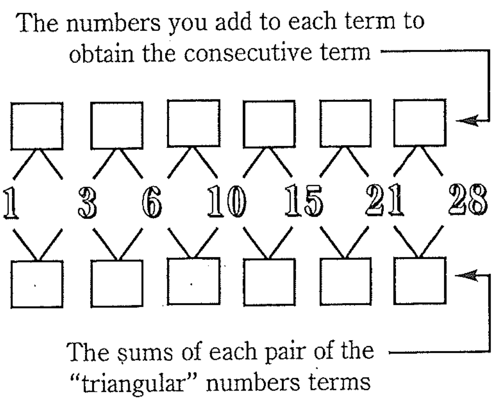


Sequence 2

Can you now answer these questions?

1. In sequence 1, how is each successive term obtained? That is, what must be *added* to each *preceding* term to obtain the subsequent term?
2. How is each successive term obtained in sequence 2?
3. What relationship is there between sequence1 and sequence 2? The illustration below may help you.





1	+	.....	=	4
3	+	.....	=	9
6	+	.....	=	16
10	+	.....	=	25
15	+	.....	=	36
21	+	.....	=	49
28	+	.....	=	64

4. Fill in missing numbers of this table that shows on the far left the sequence of triangular numbers (Sequence 1 above) and on the far right the sequence of squares (Sequence 2 above). Write a brief description of your observations.

Unit 3 End Games: Pascal's triangle revisited ★

Answer the questions.

Figure 1

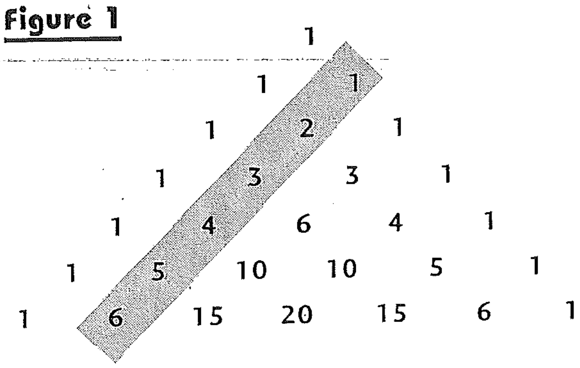
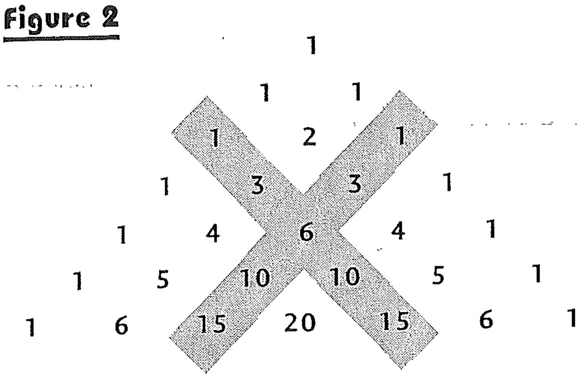


Figure 2



1. Describe the pattern formed by the shaded diagonal row.
2. Add the first two numbers: ____
   Add the second and third numbers: ____
   Add the third and fourth number: ____
   Add the fourth and fifth numbers: ____
   What is the result?
3. Describe the pattern formed by the two shaded diagonal rows. Do the numbers form a sequence? (See the preceding game.)
4. Add the first and second numbers: ____
   Add the second and third numbers: ____
   Add the third and fourth numbers: ____
   Add the fourth and fifth numbers: ____
   What is the result?

Unit 3: Final review and recall exercises.

Do you remember everything? Now is the time to check.

Exercise 1. Fill in the blanks with appropriate terms OR answer the question. ★

① $\frac{a}{b}$ is a ____________________

② $\frac{a}{b} = \frac{x}{y}$ is a ____________________

③ ay are the _______ and bx are the ________

④ In $xy = 4$, x and y ____________________.
As x increases, y ____________.

⑤ In $M = \frac{1}{3}x^2y$, M ____________________.

⑥ In $y = 2x^3$, x and y ____________________;
y varies directly ____________________.

Exercise 2. Fill in the blanks with appropriate terms. ★

14, 11, 8, 5, 2 ...

is an example of ________________________. 14, 11, and 8 are the first three ____________ of this progression. Each ________________ term is obtained by ____ preceding number. This value is called ____________________________. Algebraically this may be represented by $n - 3$, in which n is ________________ and $- 3$ is ____________. Returning to 14, 11, and 8, 11 is the ____________ ____________________. It is also the ______________ of 14 and 8.

Exercise 3. Fill in the blanks with appropriate terms. ★

| 4, 12, 36, 108 ... |

is an example of_____________________. In this progression each new succeeding member is found by _______________ the _______________ by a given value called __________________.

Exercise 4. Describe each sequence shown algebraically here. ★★

1. $(n + 5)$ ____
2. $(n + n+5)$ ____
3. $(n \cdot 3)$ ____
4. $(4n)$ ____

Exercise 5. Discussion of the geometric mean. ★★★

| 4, 12, 36, 108 ... |

Taking this progression as an example, and taking the two consecutive terms 12 and 36 as an example, describe how we can obtain the geometric mean (the mean proportional) between these two numbers. ________________________________

____

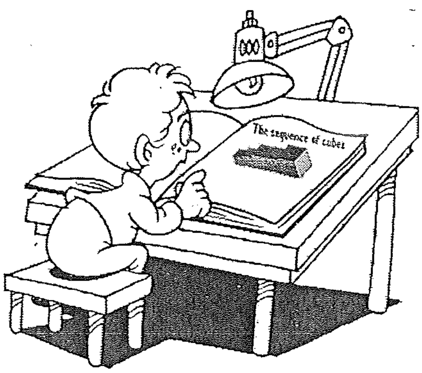
____

THE LANGUAGE OF MATHEMATICS
Summary of all important language

Proportions and variations

proportion
proportional (to)
similar to
ratio
numerical value
like: a like quantity
alike: two values are alike
colon (:)
member(s) of an equation
the means
the extremes
mean proportional
third proportional

variation(s)
to vary
direct variation
to vary directly
inverse variation
to vary inversely
joint variation
to vary jointly
a constant
to remain constant



Sequences (progressions)

a progression = a sequence
arithmetic sequence (progression)
common difference
arithmetic mean
a term (= member) of a sequence
consecutive nonconsecutive
successive term(s)
geometric progression
common ratio
mean proportional
geometric mean
binary sequence
sequence of squares
triangular numbers
correlation /to correlate with
a pattern
digital root

ANSWERS TO EXERCISES

UNIT ONE: the realm of numbers

Complete: Page 5. 1.4. 1, 2, 3, 4, and 6 are factors of 12.

Ex. 1.
1. The factors of ten are 1, 2 and 5.
2. 13 has only two factors: 1 and itself. (It is a prime number.)
3. The factors of 100 are 1, 2, 5, 10, 20 and 50.

Ex. 2. (page 5)
1. $2a(a - 3a^3$
2. $5x(3x^3 - 2x + 1)$
3. $5m(m - 2)$

Ex. 3. (page 6)
1. 2+2=4; 2x2=4, 4+0=4; 5-1=4; 7-2=4; 20/5=4; 40/10=4; 100/2=50/2=25-21=4 are just a few.
2. They are odd (and also prime numbers).
3. They are even, each is divisible by 2 (and itself).
4. 12 cannot be divided evenly by 11, but can be divided evenly by 6.
5. Because of the distributive rule.
6. As explained on page 7, addition and multiplication are commutative, associative and distributional.

1.5. Prime numbers: QUESTIONS (page 7)
1. Yes.
2. No. All odd numbers are not prime numbers. Consider: 9, 15, 21, and 27.
3. Because 14 is divisible by 2 and by 7, and prime numbers are only divisible by 1 and themselves. (have only 1 and themselves as factors).
4. True. (as explained above)

Think and Talk exercise (page 7)
The formula will not give (yield) *all* the prime numbers.

Page 8: Complex numbers. The coefficient of i can be assumed to be 1, a real number.

Ex. 4. DON'T FORGET
1. The *multiplier* is the number that you multiply (the *multiplicand*) by to get your *product*.
2. The *divisor* is the number you need to divide (*the dividend*) by to get your *quotient*.
3. *The subtrahend* is the number you subtract (from the *minuend*) to obtain the remainder correct in this exercise. Note that the remainder is also called the *difference*.
4. The mistake was: (7x4)+2= 30, instead of 7(6) which yields 42.
5. The mistake: 6 + (4x3) -2 = 16, instead of 6 + 4(3-2) = 6 + 4(1) = 12.

CREATURE FEATURE (page 10):
Soho calculated like this: 6+(4x6) -1 = 30-1 = 29.
Bozo did it like this: 6+4=10x(6-1) = 50.
Dodo did it like this: (6+4)(6) -1= 10(6)-1 = 60-1= 59.
Curly did it correctly: 6+ 4(6-1) = 6 + 4(5) = 6 + 20 = 26.

Ex. 5. (page 13)
1. the dividend; 2. the divisor; 3. the quotient.
4. terms
5. multiplicand; 6. the multiplier; 7. the product
8. of the associative, commutative and distributive laws

THINK AND TALK EXERCISE (page 14)
1. True. When both numerator and denominator have no common integral factor except 1 the fraction is in lowest terms.
2. False. We do so by dividing both numerator and denominator by their common factors.
3. False. Simply invert the divisor and multiply.
4. True.

READ AND UNDERSTAND, page 14
A: complex fraction; B: fractional equation; C: fractional equation; D: mixed expression; E: complex fraction (because the unknown is in the denominator)

1.10. A NEW PROBLEM FOR YOU (page 16)
Let x equal the number. Now write the equation: $1/2\,x = 1/3x+17$ (one-half x equals one third x plus seventeen).
Subtract $1/3x$ from the right-hand side of the equation to bring the unknown to the left-hand side: $1/2x - 1/3x = 17$.
Change the two fractions to equal fractions with common denominators, obtaining: $3/6x - 2/6x=17$.
Perform the subtraction, obtaining: $1/6x=17$.
Multiply both sides of the equation by 6 to eliminate the fraction: x=102.
Proof:
1/2(102) = 51.
1/3(102) = 34
Does 51 = 34 + 17? Yes.

1.11. MATH SYMBOLS (page 17)
1. f / 2. d / 3. e / 4. g / 5. b / 6. h / 7. c / 8. p / 9. o / 10. y / 11. z / 12. j / 13. a / 14. k / 15. aa / 16. l / 17. m / 18. x / 19. n / 20. i / 21. q / 22. r / 23. v / 24. u / 25. t / 26. s / 27. w

UNIT TWO: Powers, roots, logs and radicals

Ex 1. (page 21) 1. a to the power of minus one is 1 over a to the positive n.
2. A fraction to the power of minus one is the reciprocal of the fraction.

~~Ex. 2. (page 21) 1.~~ b to the nth power multiplied by b to the mth power is equal to b raised to the sum of the two powers, n and m. To multiply powers of the same base we add the exponents.
2. The product of ab raised to the power n is the same as the product of each factor raised to the same power.
3. a raised to the mth power , and the result then raised to the nth power is equal to a raised to the product of the two powers. We raise a power to a power by multiplying the powers: $(a^m)^n = a^{m \cdot n}$
4. x to the power minus two, plus y to the power zero
5. The product ab raised to the power minus three plus x to the power three minus y to the power zero.
6. Four (raised) to the power a minus b (raised) to the power m.

Ex. 4. TRUE OR FALSE (page 23)
1. $\sqrt{xy}$ is not equal to $\sqrt{x} + \sqrt{y}$: for instance, $\sqrt{(9)(16)} \neq \sqrt{9} + \sqrt{16}$, but it is equal to $\sqrt{9} \cdot \sqrt{16}$
2. $\sqrt{x+y}$ is not equal to $\sqrt{x} + \sqrt{y}$
3. True. $4^{1/2} = 2$.
4. Very false. $(a+b)(a+b) = a^2 + 2ab + b^2$

ANSWERS TO EXERCISES

Ex. 5. (page 24)
1. a rational number; 2. raise 2 to the third power; 3. raise two to the power four 4. an irrational number 5. a radical 6. a radical sign 7. a radicand 8. the index or order.....9. a surd

Unit 2: Final review and recall exercises (pages 26-27)
Ex 2.(page 26) 1. 3 is the index or order of the radical , or the root (here the cubic root), 8 is the radicand, √8 is the radical, and ±2 is the root (+2 being the principal cubic root of 8.)
2. The log of 10,000 to the base 10 is 4.
3. x to the power 1/2 is equal to the principal square root of x.
4. xm/n means the mth power of the nth root, or the nth root of the mth power.

Ex. 3. (page 27) 1. The square root of 125 is an irrational number the value of which to four decimal places is 11.1803. The cubic root of 125 is 5, a rational number.
2. The square root of -4 is an imaginary number. The cubic root of -8 is -2.

Ex. 4. (page 27)1. The radicand is factored into two factors (5√36 and √2x), one of which is the greater perfect power (here: 36) of the same degree as the radical. We next take the indicated root of this factor (6) which now becomes a coefficient of the resulting surd.
2. We simplify the radical with a fractional radicand by first multiplying both terms of the fraction by 2 to make the denominator a perfect square (16). We transform the single-fraction radicand into a fraction with a radical in both numerator and denominator, since √4/16 = √4/√16. Finally we take the root of the denominator: 4.

<u>UNIT THREE:</u> Proportions, ratios and sequences
Ex 1. (page 29) 1. The two polygons are similar: bcde ~b'c'd'e'. The corresponding sides of similar polygons are proportional. c over c prime is proportional to d over d prime, which is proportional to e over e prime and so on.
2. The areas of two similar polygons are proportional to the squares of their corresponding sides.
3. Triangle mon is not similar to triangle acb. Therefore, sides m, n, and o are not proportional to sides a, b and c.
4. We know that $ad=bc$ because in any proportion, the product of the means equals the product of the extremes.

Exercise 2(page 31)
1. E varies inversely as the product of 4, pi, and the square of d. As d increases, E decreases.
2. The expressions 4p are canceled out. In a fraction, the same number in the numerator and denominator is equal to 1.
3. Y increases as x increases and the two are in direct variation. 1/2 is a constant.
4. y decreases as x increases because the constant is negative.
5. xy=-7. inverse variation: y decreases as x increases. 2y=x: direct variation y increases as x increases.

Write out the following (page 31):
1. V=1/P 2. $d = t^2$ 3. $S=x4/\sqrt{y}$ 4. $M = p^2\sqrt{v}$ 5. $x_1/x_2 = m/n$ 6. $C = \Delta Q/\Delta T$ 7
7. $P^2 = a$. 8. $F=mv^2/r$ 9. $A = 1/3(x^2/y^2)$

3.4. (page 32)A. Arithmetic progressions: the coefficient of d in any given case is always **one less than the number of the term.**
Arithmetic means: the arithmetic mean is always the **average** of the two nonconsecutive numbers.Consider: 5, 7, 9: 7 is the arithmetic mean and is the average of 5 +9=14/2.

B.Geometric progressions (page 32):
Can you answer these?
1. Each successive term is obtained by multiplying by 2..
How does 4 relate to 2? $4 = 2^2$
How does 8 relate to 2? $8 = 2^3$
How does 16 relate to 2? $16 = 2^4$
How does 32 relate to 2? $32 = 2^5$
(The common ratio is thus **an exponent.**)
let a be the first term; let r be the common ratio, let l be the last term.
The first term is a.
The second term is ar.
The third term is ar^2
The fourth term is ar^3.
The fifth term is ar^4.
The nth term is ar^{n-1}.
$l = ar^{n-1}$
What observation is possible? The exponent of r in any given term is always **one less than the number of the term.**

C. Binary sequence (page 33)
The pattern shows that every counting number on the left can be expressed by one term, or by the sum of several terms, of the binary sequence shown on the right.
This is of obvious importance in the **binary system:** the binary sequence finds use in transcribing numbers and instructions used in computers and electronics in which all information is in **base 2** or **binary numerals.**

D. Sequence of squares (page 34)
RECALL: (1.) 1,3,5,17,19,21, 101 are examples of **odd** numbers; 2,4,6,18,22,66,132 are examples of **even** numbers.
(2.) 2,3,4,5,6,7,8,9 are **consecutive** numbers; 2,4,7,9,11,14 are **nonconsecutive** numbers.
(3.) 25 is the square of 5; 5 is the square root of 25.

Step 1. The sums of consecutive odd numbers beginning with 1 are **squares.** When the consecutive odd numbers are **added,** their sum is a number (**square**) whose **square root** is equal to the number of **digits/terms** in the original sequence of odd numbers. From the pattern above we can predict that $1+3+5+7+9+11+13+17 = 9^2$ (nine being the number of terms in the sequence).

Step 2. The new number is the **digital root of the square.** For example: 8: The square is 64. Add the two digits: 6+4 to get 10, another two-digit number. Now add the two digits of this last number: 1+0 to get 1, the digital root of 8. The digital roots of 1-8 form a **palindrome** (The same sequence is obtained reading forward or backward).

ANSWERS TO EXERCISES

E. Read and understand (page 34):The Fibonacci sequence 1 1 2 3 5 8 13.21 34 ... Every third term is evenly divided by 2.

3.5. Pascal's triangle

Question: The numbers in a given row correspond to the numerical coefficients in the binomial expansion, with each row corresponding to the integral power to which the binomial is to be raised.

Binomial formula (page 36): The 2nd term: $4x^3y/1$; the 3rd term: $12x^2y^2/2!$; the 4th term: $24xy^3/3!$; the 5th term: $24y^4/4!$

Think and Talk exercise (page 36): 1. The exponent of y is one less than the number of the term. (Note: y does not appear in the first term).

2. The exponent of x is n minus the exponent of y for that same term.

3. The number of factors in both the numerator and denominator of the coefficient is the same as the exponent y for that same term.

UNIT THREE: End Games: Sequences

The first 7 terms of sequence 1: 1 3 6 10 15 21 28
The first 7 terms of sequence 2: 2 4 9 16 25 36 49

1. The successive term is obtained by adding the next counting number as shown in the illustration: 1+*2* = 3; 3 + *3* = 6; 6 + *4* = 10 and so on.

2. In sequence 2, each successive term is obtained by adding the next highest odd number: 1+*3*=4; 4+*5*=9; 9+*7*=16 and so on, forming the sequence of squares (square numbers).

3. To any given term of sequence 1, add the next term in that same sequence to obtain the next term in the sequence of squares.

4. To the first "triangular" number is added the next highest "triangular number" (1+3) to obtain a term in the sequence of squares. (1+3=4; 3+6=9, and so on)

UNIT THREE: End Games: Pascal's triangle revisited (page 38) Figure 1. (1) The pattern: the consecutive counting numbers. (2) The sums: 3 5 7 9 11; the pattern: the consecutive odd numbers.

Figure 2. (3) The pattern: the triangular numbers seen in the preceding End Game. (4) The sums: 4 9 16 25; the pattern: sequence of squares

UNIT THREE: Final review and recall exercises

Ex. 1. (page 38) 1. a ratio (and a fraction, of course). 2. a proportion (and an equation, too). 3. ay are the extremes, bx the means. 4. x and y vary directly. As x increases, y decreases.

5. M varies jointly as x and y.

6. x and y are in direct variation. y varies directly as twice the cube of x.

Ex. 2. (page 38) An example of an arithmetic sequence. 14, 11 and 8 are the first three terms (members). Each succeeding/successive term is obtained by subtracting 3 from the preceding number. This is the common difference. n is any number in the series, -3 is the common difference. 11 is the arithmetic mean between 14 and 8 and is also the average of 14 and 8.

Ex. 3 (page 39) 4 12 36 108 is an example of a geometric sequence (progression). In this progression each new successive member is found by multiplying the preceding term by a value called the common ratio (here: 3).

Ex. 4. (page 39) 1. The common difference is +5: add 5 to a given term to obtain the successive term.

2. To any given term add the sum of the value of that term and 5. (2, 9, 23, 51 and so on).

3. multiply the term by 3 to obtain the succeeding term (thus, a geometric progression).

4. multiply the term by 4

Ex. 5 (page 39)

The geometric mean, or mean proportional, is the square root of the product of the two consecutive terms, or as stated earlier, is the length of the side of a square whose area is equal to the product of the two numbers. This is fine when the two terms are 5 and 125 for the product will be 625 (the area of the square) and the side will be the square root of 625 or ±25. But in this case the product of 12 and 36 is 432 and the square root: 20.7846.

UNIT FOUR: Permutations, Combinations & Probability

Application exercise: throwing a die, page 41.

The chances of your number coming up are 1/6. The probability is one out of six, or one over six, which works out to approximately 18%. The odds against your number coming up are 5/6, or approximately 83%.

Talk and think exercise: Think it out, say it out loud.

Pasha and the fish, page 41.

1. The total number of combinations of 12 fish taken 2 at a tile is given by 12x11/2! which comes out to 66.

2. The total number of ways of combining 1 red fish with one white fish is 7x5 which comes out to 35.

3. The number of possible favorable events, or the chances of having 1 red fish and 1 white fish are 35/66 or approximately 53%.

4. The odds against having one red and one white fish are 31/66 (thirty-one to sixty-six) which works out to approximately 47%.

Check that you understand, page 42.

1. $p = y/x+y$: this expresses the probability that the event will not happen, that is, the odds against it happening. If x=0, this means that the event is certain not to happen.

2. The probability of a favorable event is the ratio of the total number of possible favorable events over the total number of all events, both favorable and unfavorable. If x=0, the probability of a favorable event becomes 1, that is, it is certain to happen.

3. This equation states that the number of favorable events added to the total number of unfavorable events must be equal to 1. Or, equivalently, the probability that something will happen + the probability that it won't happen is equal to 1.

UNIT FIVE: Functions and their graphs

Think and Talk exercise.

1. True. 2. True.

Exercise 1.

1. first degree (in x) 2. second degree in y 3. first

ANSWERS TO EXERCISES

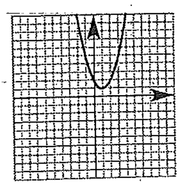
degree in x 4. second degree in x and y 5. third degree in x and y 6. fifth degree in x and y.

Exercise 2.
1. a first degree function of x; 2. first degree function of x; 3. second degree function of x and y; 4. third degree function of x and y; 5. second degree function of x and y; 6. third degree function of x and y; 7. fifth degree function of x and y; 8. third degree function of x and y; 9. fourth degree function of x

Exercise 3.
1. For *any* value which we *assign* to x there is always a *corresponding* value of y, and *conversely.*
2. A function may be defined as a *relationship* pairing two *sets* of numbers *such that* to each number in the first *set* there *corresponds* one and only one number in the second *set.*
3. A function exists when two *variables* are so *related* that for each value of one *variable(the independent variable)* there is exactly one value of the other *variable (the dependent variable).*
4. The first expression is *rational,* while the second is not because the exponent is not a positive integer.
5. The first is an *integral expression,* while the second is not because it contains the unknown in the denominator.

Definition: page 50
The y-intercept of a linear graph is the distance from the origin, 0, to the point where the graph of the line intersects the y-axis.

5.4. Section B: Other functions with curved graphs.
Hyperbolas: As x approaches 0, y approaches infinity. x and y are variables. They vary inversely.
Ellipses: In the equation above, for any value of $x>5$, y will be imaginary.
Definition: An ellipse is a locus of points the sum of whose distances from two given points (the foci) is constant.
5.4. Functions and their graphs (page 53).
The graphed function should look like this:



x= -2, y=10
x=-1, y= 5
x= 0, y= 2
x= +1, y= 1
x=+2, y= 2
x=+5, y=10

5.4. Fill in the description:

Point Q is the turning point of the curve. At Q the function changes from a decreasing function to an increasing function. The coordinate -2¼ represents the minimum value of the function.

This parabola is concave upward or U-shaped as opposed to the parabola shown in the next figure. Parabolas will have this shape when the coefficient (of x) is positive.

The parabola in figure 2 is concave downward. A parabola will have this shape when the coefficient of x is negative.

Point P represents the turning point.

B. Other functions with curved graphs: page 54

B1: As x approaches O, y approaches infinity (since a denominator of zero is undefined). x and y in the equation $xy=c$, where c is a constant, vary inversely.

B2: For any value $x>5$, y will be imaginary.

Definition: An ellipse is the locus of points the sum of whose distances from two given points (the foci) is a constant.

Think and Talk: page 54.
1. If $c=0$, the resulting graph will be two intersecting straight lines.
2. If $xy=0$ the two intersecting straight lines coincide with the x- and y-axes, respectively.
3. If $a=b$, then the resulting geometric figure will be a circle, not an ellipse.
4. A. A circle results from the equation $x2 + y2=r2$. A circle is a locus of points at a given distance (OP) from a given point (here: O, the origin). The formula may be derived by applying the Pythagorean theorem: OB$=x$; $BP=y$ and $OP = r$ (the radius).
B. An ellipse is the locus of points the sum of whose distances from two foci remains constant. If P is any point of the ellipse, then $PF + PF' = k$ (a constant).
C. A hyperbola is the locus of points the difference of whose distance from two given points (the foci) is constant. If P is any point on either branch of the hyperbola, then $PF - PF' = k$ (a constant)

Exercise 6
2. the limit of a root is the root of the limit
3. the limit of a sum is the sum of the limits
4. the limit of a difference is the difference of the limits
5. the limit of a product is the product of the limits.
6. Here's how to write *the limit of a quotient is the quotient of the limit*:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$$

Exercise 7
Function #1: differentiable
Function #2: non differentiable, $x=1$ tangent line is vertical
Function #3: non differentiable, a discontinuous function
Function #4: non differentiable at $x=-2$ (discontinuous).

Think and talk exercise, page 58.
If we attempt to integrate x^{-1}, applying the power rule, we will have 0 in the denominator (1/0) and division by zero is undefined.