Université de Haute-Alsace
IRIMAS, Département de Mathématiques

Thèse

pour obtenir le grade de
Docteur en Mathématiques

présentée par

Francisco Chaves

Holonomy and equivalence of analytic foliations

M	Frank Loray	Université de Rennes I	(Rapporteur)
Mme	Helena Reis	Université de Porto	(Rapporteur)
M	Dominique Cerveau	Université de Rennes 1,	(Examinateur)
M	Yohann Genzmer	Institut de Mathématiques de Toulouse	(Examinateur)
M	Daniel Panazzolo	Université de Haute-Alsace	(Directeur)

Remerciements

Parler de personnes importantes à cette époque en quelques pages est une tâche aussi difficile qu'écrire la thèse elle-même.

D'abord, je tiens à remercier le Pr. Daniel Panazzolo qui était avec moi à travers toutes les étapes de ma thèse à m'instruire et à me corriger. Pour avoir été patient avec les choses que je ne comprenais pas et pour avoir toujours été didactique et organisé dans nos conversations. Je vais certainement employer ce que vous m'avez appris durant ces années pendant longtemps. Je suis très reconnaissant aux professeurs Frank Loray et Helena Reis d'avoir accepté d'être rapporteurs de ce travail, sans leurs travaux cette thèse n'existerait pas. Merci également aux professeurs Dominique Cerveau Yohann Genzmer de faire partie de ce jury.

Chronologiquement, je tiens à remercier le Pr. Alexandre d'avoir eu confiance en ma façon de travailler et de m'avoir recommandé pour ce poste. Peu de temps après, il y a toutes les personnes qui m'ont soutenu émotionnellement pour faire face au changement de pays et surtout à l'éloignement de mes garçons. Pour n'en citer que quelques-uns, je remercie tous mes collègues de Master, particulièrement Roger. Grâce à d'innombrables entraînements à l'oral, j'ai réussi à atteindre un bon niveau pour la soutenance de ma maîtrise.

En ce qui concerne mes enfants, je dois remercier Rafaela pour tout le temps qu'elle a eu à s'occuper d'eux du mieux qu'elle a pu. Un grand merci à ma sœur qui était souvent tante, mère et père. Leur présence auprès d'eux m'a permis de me détendre, sachant qu'il y avait quelqu'un de prêt à faire tout ce qu'il fallait. À mon frère, merci pour tes conseils sur la façon d'agir et pour avoir, toujours dit la vérité, aussi dure soit-elle. Je te souhaite tout le meilleur pour l'avenir.

À Mulhouse, et même avant ça, je remercie Hamilton pour tout, et ici je devrais écrire TOUT. Pour calmer ma mère, pour m'avoir expliqué les différents papiers, pour venir me chercher en Suisse, me montrer Mulhouse, etc. Je pourrais passer des heures et des heures à lister tout ce qu'il a fait pour moi. Aninda et les cacahuètes vous étiez mon groupe à Mulhouse. Les soirées n'auraient pas été si intéressantes sans vous. Toutes les blagues, les conversations, les déceptions amoureuses, tout était remarquable. Même les chimistes n'avaient pas prédit que des sujets aussi distincts généreraient un mélange aussi homogène.

Pendant la période de confinement, pendant presque un an, j'ai été confiné dans 16m^2. Ce fut sans aucun doute un moment de grande réflexion. Je dois beaucoup à Beatriz pour toute sa présence. Souvent même sans parler, mais là, sur l'écran. Pour les films que nous avons regardés ensemble et pour toutes les milliers d'histoires et de plaintes que je t'ai racontées. Je pense que je devrais remercier Mark Zuckerberg pour cela aussi.

Le laboratoire de nouveau réuni, les après-midi étaient plus légères avec la présence de Thomas et Quentin. Le Shamrock a de nouveau pris sens avec vous. Pourquoi pas un shot de schnaps ? Venant de Mulhouse, je voudrais citer Marouane et Pablo, des amis que la salle de sport m'a amenés. Pour les dimanches après-midi à toujours essayer de construire une équipe avec Momo, même si cela ne fonctionne presque jamais. Mulhouse est plus amusant avec vous les gars.

Enfin, merci à M. Makhlouf et Mme Kuhn pour tout le soutien administratif et toujours le sourire aux lèvres pour mes questions, ainsi que tout le personnel de l'université qui m'a aidé à un moment ou à un autre.

Contents

Introduction

The analytic classification of singular analytic foliations in dimension two and its connection with the analytic conjugation of the corresponding holonomies was one of the central results of the well-known paper of Mattei and Moussu [MM80b] in 1980.

Later in 1984, Elizarov and Il'Yashenko [EI84] proved that, in dimension three or greater, if we add some restrictions on the vector field that generates the foliations, the analytic conjugation of the holonomies corresponds to the analytic equivalence of the foliations. In the year 2006, Helena Reis [Rei06] reproved the same result, but with a simpler method.

In more details, the authors consider germs of singular analytic vector fields X in $(\mathbb{C}^n, \mathbf{0})$ for $n \geq 3$, with $\lambda_1, \ldots, \lambda_n$ as the eigenvalues of the linear part of X, verifying:

1. The origin of $\mathbb{C}^n$ is an isolated singularity of X.
2. X is of Siegel type (i.e, the convex hull of $\lambda_1, \ldots, \lambda_n$ contains the origin in $\mathbb{C}$).
3. All the eigenvalues are nonzero and there exists a straight line through the origin of $\mathbb{C}$ separating λ_1 from the others eigenvalues in the complex plane.
4. Up to a change of coordinates, $X = \sum_{i=1}^{n} \lambda_i z_i (1 + f_i(z)) \partial_{z_i}$, where $z = (z_1, \ldots, z_n)$, and f_i is a germ of analytic function such that $f_i(0) = 0$ for all i.

In [EI84] and [Rei06], it is proved the following.

Theorem ([Rei06], Theorem 1)**.** *Let X and Y be two germ vector fields, verifying $(1), (2), (3)$ and (4). Denote by Δ_X and Δ_Y the holonomies of X and Y relatively to the separatrices of X and Y tangent to the eigenspace associated with the first eigenvalue, respectively. Then if Δ_X and Δ_Y are analytically conjugated, X and Y are orbitally analytically equivalent.*

In this work, we drop the hypothesis (1), and weaken $(2), (3)$, and (4). As consequence, we enlarge the set of vector fields for which the equivalence between the conjugations of the holonomies and analytic equivalence of the foliations holds.

More precisely, we treat a class of germs of singular analytic foliations called crossing type. A *crossing type* foliation in $(\mathbb{C}^{n+1}, 0)$ is a triple $(\mathcal{F}, H, \Gamma)$ such that:

i. $\mathcal{F}$ is a germ of 1-dimensional analytic foliation.

ii. H is a smooth hyper-surface and Γ is a smooth curve such that:

    (a) H and Γ are transverse at the origin.

    (b) Both are invariant by the foliation $\mathcal{F}$.

iii. Each local generator of $\mathcal{F}$ has a nonzero eigenvalue in the Γ-direction.

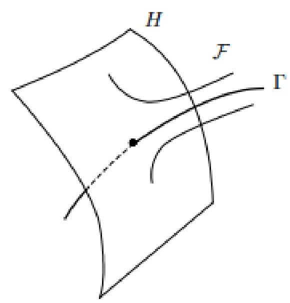


Figure 1: Transversality of H and Γ at origin

As in the papers cited above, we have to demand a property on the eigenvalues of the linear part of the local generators. We say that a vector field, with $1, \mu_1 \dots, \mu_n$ as the eigenvalues of its linear part, has *no transverse negative resonance* if no element in the positive cone

$$\mathcal{C} = \left\{ \sum_{i=1}^{n} p_i \mu_i ; p_i \in \mathbb{Z}_{\geq 0}, p_1 + \cdots + p_n \geq 1 \right\}$$

can be written in the form $\mu_j + q$, with $q \in \mathbb{Z}_{\geq 1}$, for any $1 \leq j \leq n$, see Figure 2.

As consequence of the definition of crossing type foliation, there exist local coordinates $(x, \mathbf{z})$, so-called adapted to $(\mathcal{F}, H, \Gamma)$, such that the hypersurface H and the curve Γ are expressed by $H := \{x = 0\}$ and $\Gamma := \{\mathbf{z} = 0\}$. Moreover, if the local generators of $(\mathcal{F}, H, \Gamma)$ have no transverse negative resonance, we can choose a local generator in these adapted coordinates which has the form

$$x\partial_x + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} z_j \partial_{z_i} + \sum_{i=1}^{n} b_i(x, \mathbf{z}) \partial_{z_i}, \tag{1}$$

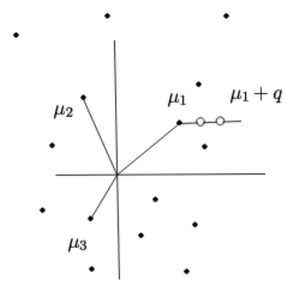


Figure 2: No transverse negative resonance

where $(a_{ij})_{n\times n}$ is a constant matrix, and $b_i(x,0) = \frac{\partial b_i}{\partial z_j}(x,0) = 0$ for all $i,j \in \{1,\ldots,n\}$. We say that a vector field of this form is an *x-normalized* vector field.

Our main goal is to classify such singular foliations up to analytic equivalence. Here, we say that two crossing type foliations $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ are *analytically equivalent* if there exists an analytic change of coordinates mapping the leaves of $\mathcal{F}$ to the leaves of $\mathcal{G}$ and the pair (H,Γ) to (L,Ω).

The following theorem is the main result in this work. We adapted and generalized an idea originally introduced in the thesis of Arame Diaw [Dia19, DL20] to prove it. Below, we denote by Γ-*holonomy* and Ω-*holonomy* the respective local holonomies along the curves Γ of $\mathcal{F}$ and Ω of $\mathcal{G}$.

Main Theorem. *Let $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ be two crossing type foliations such that:*

(a) The linear part of the local generators of $\mathcal{F}$ and $\mathcal{G}$ are conjugated.

(b) The local generator of $\mathcal{F}$ (and therefore of $\mathcal{G}$) has no transverse negative resonance.

(c) The respective Γ-holonomy and Ω-holonomy are analytically conjugated.

Then $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ are analytically equivalent.

Motivated by the Main Theorem, we say that a crossing type foliation $(\mathcal{F}, H, \Gamma)$ is *analytically classified by its linear part and its Γ-holonomy* if all crossing type foliations $(\mathcal{G}, L, \Omega)$ with a conjugated linear part and a conjugated Ω-holonomy to $(\mathcal{F}, H, \Gamma)$ is analytically equivalent to $(\mathcal{F}, H, \Gamma)$.

As a consequence of this theorem, we can give a unified proof for a result obtained with different tools by Mattei and Moussu [MM80b] and later by Martinet and Ramis [MR82]. In a recent paper [DL20], Diaw and Loray also use similar techniques to reprove this theorem. In our notation, we can enunciate it as follows.

Corollary A. *Consider a crossing type foliation $(\mathcal{F}, H, \Gamma) \in (\mathbb{C}^2, 0)$ which has, in adapted coordinates, an x-normalized local generator of the form*

$$X = x\partial_x + yf(x, y)\partial_y,$$

where f is a germ analytic function such that $f(0, 0) = \lambda$. Then two cases can occur.

1. *The eigenvalue λ belongs to $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then $(\mathcal{F}, H, \Gamma)$ is analytically linearizable.*

2. *The eigenvalue λ belongs to $\mathbb{R}_{\leq 0}$, then $(\mathcal{F}, H, \Gamma)$ is analytically classified by its linear part and its Γ-holonomy.*

We recall that a germ of singular vector field in $(\mathbb{C}^n, \mathbf{0})$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ is in the Siegel (resp. Poincaré) domain if the origin lies (resp. does not lie) in the convex hull of the eigenvalues in $\mathbb{C}$.

Proof. In the Poincaré case ($\lambda \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$), the result is immediate since the existence of two analytic separatrices implies that there can be no resonance of Poincaré type. In the Siegel case ($\lambda \in \mathbb{R}_{\leq 0}$), it is sufficient to remark that the condition of no transverse negative resonance is satisfied. Hence, the result is a consequence of the Main Theorem. □

In dimension three, the following corollary, which will be proved in section 5.3, is a generalization of this result.

Corollary B. *Consider a crossing type foliation $(\mathcal{F}, H, \Gamma) \in (\mathbb{C}^3, 0)$ which has, in adapted coordinates, an x-normalized local generator with semi-simple part*

$$x\partial_x + \lambda y\partial_y + \mu z\partial_z.$$

Then three cases can appear:

1. *The eigenvalues are in the Poincaré domain. Then $(\mathcal{F}, H, \Gamma)$ is analytically normalizable and has at most a finite number of resonant monomials.*

2. *The eigenvalues are in the Siegel domain and at least one of the eigenvalues λ, μ is non-real. Then $(\mathcal{F}, H, \Gamma)$ is analytically classified by its linear part and its Γ-holonomy.*

3. *The eigenvalues are in the Siegel domain and all real. Then either $(\mathcal{F}, H, \Gamma)$ is analytically classified by its linear part and its holonomy or one of the following conditions holds up to a permutation of the y and z coordinates:*

   (a) *Either $\mu < \lambda \leq 0$ and*
   $$p\lambda = \mu + q, \tag{2}$$
   for some $p, q \in \mathbb{Z}_{\geq 1}$.

   (b) *Or, $\mu \leq 0 < \lambda$, and either (2) holds or $\lambda \in \mathbb{Q}_{>0} - \mu\mathbb{Q}_{\geq 0}$ (notice that these conditions are not mutually exclusive).*

In higher dimensions, a similar list of possible cases is considerably more complicated. In the last section, we are going to prove a similar result in dimension 4, with the additional assumption that the linear part of the local generator is a real matrix. The statement is quite technical and we refer to Section 5.6 for the details.

The necessity of the no transverse negative resonance condition in the Main Theorem is a natural question. In this direction, Proposition C below gives a recipe to construct examples where this condition is violated, and we obtain non-equivalent crossing type foliations with conjugated holonomies.

Consider a diagonal vector field $S = x\,\partial_{\mathrm{x}} + L(\mu)$, where $L(\mu) = \sum_{i=1}^{n} \mu_i z_i\,\partial_{\mathrm{z_i}}$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$. The *negative and positive resonant sets* of S are given by

$$\begin{aligned}\mathrm{NR}(S) &:= \{T \in \mathcal{L}_n; \langle \mu, T\rangle \in \mathbb{Z}_{\geq 1}\},\\ \mathrm{PR}(S) &:= \{K \in \mathcal{L}_n \setminus \{\mathbf{0}\}; \langle \mu, K\rangle \in \mathbb{Z}_{\leq 0}\},\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{C}^n$, and we denote by $\mathcal{L}_n$ the set of n-uples $K \in \{(\mathbb{N}^n - e_1) \cup \cdots \cup (\mathbb{N}^n - e_n)\}$, such that $|K| \geq 0$. In other words, we consider the set of n-uples $K = (k_1, \ldots, k_n)$ of integer numbers such that at most one entry is negative (and equal to -1) and the sum $k_1 + \cdots + k_n$ is greater equal zero.

The following result will be proved in Section 5.5.

Proposition C. *Let $\mu = (\mu_1, \ldots, \mu_n)$ be a n-uple in $\mathbb{C}^n$ such that $\mu_i \neq 1$ and $\mu_i \neq \mu_j$. Suppose that there exist a pair $(T, K) \in \mathrm{NR}(S) \times \mathrm{PR}(S)$ and a nonzero vector $\lambda \in \mathbb{C}^n$ such that*

1. $\langle \mu, T\rangle + \langle \mu, K\rangle \leq 0$,
2. *the vector field $\mathbf{z}^T L(\lambda)$ is analytic, and*
3. $\langle \lambda, K\rangle = 0$.

Then there exist two x-normalized analytic vector fields X, Y with semi-simple part S that generate analytic foliations which have conjugated holonomies along the x-axis but are not orbitally analytically equivalent.

As an application, we obtain the following explicit example in dimension 3.

Exemple 1. *The vector fields*

$$\begin{aligned}X &= x\,\partial_{\mathrm{x}} + 2(1 + x^2 z)(y\,\partial_{\mathrm{y}} - z\,\partial_{\mathrm{z}}) \text{ and}\\ Y &= x\,\partial_{\mathrm{x}} + 2(1 + x^2 z)(y\,\partial_{\mathrm{y}} - z\,\partial_{\mathrm{z}}) - 2y^2 z\,\partial_{\mathrm{y}}\end{aligned}$$

generate foliations with conjugated holonomies along the x-axis but are not orbitally analytically equivalent.

Notice that the above example belongs precisely to the special case (3.b) of Corollary B, with $\mu = -2$ and $\lambda = 2$.

In the light of the above example, we recall the well-known fact that the holonomy computed along a *weak separatrix* does not always classify a germ of a singular foliation. The saddle-node foliation in $(\mathbb{C}^2, 0)$ is probably the simplest example (see, e.g. [CCD13], Section 6.6.3). However, to our knowledge, the Example 1 is the first one illustrating that this phenomenon can also occur for the holonomy computed along a strong separatrix.

Overview of the work

In Chapter 1, we make a brief introduction to the theory of vector fields and foliations. In particular, we formalize many notations that are going to be used later. In the next chapter, the basic tool used in this work is presented in Section 2.1, the concept of $D_{r,R}$-transversely formal series. A $D_{r,R}$-*transversely formal series* is a formal series of the form

$$\sum_{k_i \in \mathbb{N}} f_K(x)\mathbf{z}^K, \tag{3}$$

where $\mathbf{z}^K = z^{k_1} \dots z^{k_n}$, and each coefficient $f_K(x)$ is convergent in the annulus $D_{r,R} := \{x \in \mathbb{C}; r < \|x\| < R\}$, where $r, R > 0$.

In Section 2.3, we develop the theory of endomorphisms in the ring of $D_{r,R}$-transversely formal series. In particular, the objective is to treat $D_{r,R}$-transversely formal derivations (derivations over the ring of the $D_{r,R}$-transversely formal series) as vector fields with coefficients being $D_{r,R}$-transversely formal series.

In Section 3.1, we construct the holonomy for a given separatrix as an application of the technique of path lifting. A brief review of previous results on analytic and formal classification of germs of singular vector fields is made in Section 3.2. In the last section of Chapter 3, we expose the work of Elizarov and Il'yashenko, Helena Reis, and Mattei and Moussu about the relations between the conjugation of holonomies and the analytic classification of foliations.

In chapter 4, we extend to $D_{r,R}$-transversely formal derivations the notions of exponential map, normal form, and symmetries. After this general study, we focus on the class of D_R-transversely formal derivations (derivations with form (3), where each coefficient $f_K(x)$ is convergent in the disk D_R) in Section 4.3.

Finally, in Chapter 5, we prove the Main Theorem and its corollaries in dimensions three and four. In Section 5.5, we study the necessity of the no transverse negative resonance property in the Main Theorem.

Index of notations

General notations

Small bold letters and numbers represent multidimensional objects

Capital letter such as X, Y, W, and Z denote vector fields

A point $p \in \mathbb{C}^{n+1}$ is denoted by $p = (x, \mathbf{z}) = (x, z_1, \dots, z_n)$

$\mathbb{N}$	the set of natural numbers with zero
A^*	the set $A \setminus \{\mathbf{0}\}$, where A is a set in $\mathbb{C}^n$
$(\mathbb{C}^n, \mathbf{0})$	a neighborhood of $\mathbf{0} \in \mathbb{C}^n$
e_i	the vector $(0, \dots, 1, \dots, 0)$ where the unique entry 1 is in the i^th position
$\lvert\cdot\rvert$	the absolute value in $\mathbb{R}$
$\lvert\cdot\rvert$	the norm in $\mathbb{Z}^n$ given by the sum of the entries
$\lVert\cdot\rVert$	the module in $\mathbb{C}$
$\mathcal{L}_n$	the set of n-uples $K \in \{(\mathbb{N}^n - e_1) \cup \dots \cup (\mathbb{N}^n - e_n)\}$, such that $\lvert K\rvert \geq 0$
$\mathcal{L}_{n,m}$	the set of n-uples $K \in \mathcal{L}_n$, such that $\lvert K\rvert = k_1 + \dots + k_n \geq m$

Chapter 1

$\mathcal{M}, \mathcal{N}$	complex manifolds
$\mathcal{T}_p\,\mathcal{M}$	the tangent space of a complex manifold $\mathcal{M}$ at a point p

$\mathcal{TM}$	‖ the tangent bundle of a complex manifold $\mathcal{M}$
$\mathbf{z}' = X(\mathbf{z})$	‖ the ordinary differential equation associated to the vector field X
$\mathrm{F_X}(t,p)$	‖ an integral curve of vector field X with initial ‖ condition $\mathrm{F_X}(0,p) = p$
$\mathcal{F}, \mathcal{G}$	‖ foliations of complex manifolds
$\mathcal{P}$	‖ a plaque of a foliation
$\mathcal{L}$	‖ a leaf of a foliation
τ	‖ a transverse section of a plaque
Δ_{τ_1,τ_2}	‖ the correspondence map between transverse ‖ sections τ_1 and τ_2
Δ_γ	‖ the holonomy map associated to a curve γ
$\mathbb{S}^1$	‖ the set $\{z \in \mathbb{C}; \|z\| = 1\}$
$\pi_1(\mathcal{M}, p)$	‖ the homotopy group at a point p in a ‖ complex manifold $\mathcal{M}$
$\mathfrak{g}$	‖ a generic group
$\mathfrak{g}^{op}$	‖ the opposite group to a group $\mathfrak{g}$
$\mathrm{Hol}(\mathcal{L}, p)$	‖ the holonomy group of a leaf $\mathcal{L}$ at a point p
$\mathrm{Hol}(\mathcal{L})$	‖ the holonomy group of a leaf $\mathcal{L}$

Chapter 2

$\mathcal{R}$	‖ a commutative ring
$\mathcal{R}[\mathbf{z}]$	‖ the ring of polynomials in the variables $z_1, \ldots, z_n$ ‖ with coefficients in $\mathcal{R}$
$\mathfrak{m}$	‖ the maximal ideal defined by $\langle z_1, \ldots, z_n \rangle$ in $\mathcal{R}[\mathbf{z}]$

$J^{(k)}$	‖ the k-jet space defined by $\mathcal{R}[\mathbf{z}]/\,\mathfrak{m}^{(k+1)}$
π_{ji}	‖ is a bounding map
$(J^{(i)}, (\pi_{ji}))_{j\leq i\in\mathbb{Z}_{\geq 0}}$	‖ an inverse system indexed by $\mathbb{Z}_{\geq 0}$ où $J^{(i)}$ is the i-jet space
$\mathcal{R}[[\mathbf{z}]]$	‖ the ring of formal power series with coefficients ‖ in a ring $\mathcal{R}$
π_k	‖ the k^{th}-truncation map
D_R	‖ the disk $\{z \in \mathbb{C};\ \|z\| < R\}$, where $R > 0$
$D_{r,R}$	‖ the annulus $\{z \in \mathbb{C}\ ;\ r < \|z\| < R\}$, where $r, R > 0$
$\mathcal{O}_R$	‖ the ring of germs of analytic functions defined ‖ in the disk D_R
$\mathcal{O}_{r,R}$	‖ the ring of germs of analytic functions defined ‖ in the annulus $D_{r,R}$
$\mathbb{C}[[\mathbf{z}]]$	‖ the ring of formal power series
$\mathcal{O}_R[[\mathbf{z}]]$	‖ the ring of D_R-transversely formal series
$\mathcal{O}_{r,R}[[\mathbf{z}]]$	‖ the ring of $D_{r,R}$-transversely formal series
$\mathcal{O}_R\{\mathbf{z}\}$	‖ the ring of D_R-transversely convergent series
$\mathcal{O}_{r,R}\{\mathbf{z}\}$	‖ the ring of $D_{r,R}$-transversely convergent series
$\sup_V \|f\|$	‖ the supremum of the function f in the set V
$\mathcal{A}(\mathbb{C}[[\mathbf{z}]])$	‖ the group of automorphisms of $\mathbb{C}[[\mathbf{z}]]$
$\mathrm{End}_{\mathbb{C}}(\mathcal{R}[[\mathbf{z}]])$	‖ the set of $\mathbb{C}$-linear endomorphisms of $\mathcal{R}[[\mathbf{z}]]$, ‖ where $\mathcal{R}$ is a commutative ring
$\mathcal{A}(\mathcal{O}_{r,R}[[\mathbf{z}]])$	‖ the group of automorphisms of $\mathcal{O}_{r,R}[[\mathbf{z}]]$
$\mathcal{A}(\mathcal{O}_R[[\mathbf{z}]])$	‖ the group of automorphisms of $\mathcal{O}_R[[\mathbf{z}]]$

$[X, Y]$	‖ the Lie bracket defined by $X \circ Y - Y \circ X$
$\mathcal{D}(\mathbb{C}[[\mathbf{z}]])$	‖ the $\mathbb{C}[[\mathbf{z}]]$-module of derivations of $\mathbb{C}[[\mathbf{z}]]$
$\mathcal{D}(\mathcal{O}_R[[\mathbf{z}]])$	‖ the $\mathcal{O}_R[[\mathbf{z}]]$-module of derivations of $\mathcal{O}_R[[\mathbf{z}]]$
$\mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$	‖ the $\mathcal{O}_{r,R}[[\mathbf{z}]]$-module of derivations of $\mathcal{O}_{r,R}[[\mathbf{z}]]$
$\mathcal{D}(\mathcal{O}_R\{\mathbf{z}\})$	‖ the $\mathcal{O}_R\{\mathbf{z}\}$-module of derivations of $\mathcal{O}_R\{\mathbf{z}\}$
$\mathcal{D}(\mathcal{O}_{r,R}\{\mathbf{z}\})$	‖ the $\mathcal{O}_{r,R}\{\mathbf{z}\}$-module of derivations of $\mathcal{O}_{r,R}\{\mathbf{z}\}$

Chapter 3

$\mathcal{K}(\lambda)$	‖ the convex hull of the set $\{\lambda_1, \ldots, \lambda_n\}$
$\mathcal{O}_{\mathcal{M},p}$	‖ the ring of germs of analytic function defined ‖ in a neighborhood of $p \in \mathcal{M}$.
Σ	‖ the singular set of a singular foliation
$X^{(k)}$	‖ the derivation induce in the k-jet space of $\mathbb{C}[[\mathbf{z}]]$ ‖ by a derivation $X \in \mathcal{D}(\mathbb{C}[[x, \mathbf{z}]])$
X_s	‖ the semisimple part of a derivation X
X_n	‖ a nilpotent part of a derivation X
$L(\lambda)$	‖ the derivation $\sum_{i=1}^{n} \lambda_i z_i \, \partial_{\mathrm{z_i}}$, where $\lambda_i \in \mathbb{C}$
ad_X	‖ the brackets operator given by $Y \mapsto [X, Y]$.
$\mathcal{S}$	‖ a separatrix of a germ of a foliation
$\mathcal{V}_n$	‖ the space of germs analytic vector fields at $\mathbf{0} \in \mathbb{C}^n$ ‖ which are singular at the origin and whose linear parts are ‖ non-degenerate with simple eigenvalues
$\mathrm{L}^{\gamma}_{\mathcal{F}}$	‖ a lift of a curve γ through the leaves of a foliation $\mathcal{F}$

Chapter 4

$\mathfrak{m}^0$	‖ a ring $\mathcal{R}$ where $\mathfrak{m}$ is the maximal ideal

$\mathcal{N}(\mathcal{O}_{r,R}[[\mathbf{z}]])$	‖ the set of $D_{r,R}$-transversely formal nilpotent derivations
$\mathcal{A}_k(\mathcal{O}_{r,R}[[\mathbf{z}]])$	‖ the set of $D_{r,R}$-transversely formal automorphisms ‖ tangent to the identity to order k
$\mathcal{N}_k(\mathcal{O}_{r,R}[[\mathbf{z}]])$	‖ the set of k-flat $D_{r,R}$-transversely formal derivations
Ad_X	‖ the map $Y \mapsto \exp X \circ Y \circ (\exp X)^{-1}$, where ‖ X and Y are derivations such that X is nilpotent
$\mathcal{A}_{norm}(\mathcal{O}_{r,R}[[\mathbf{z}]])$	‖ the set of x-normalised automorphisms of $\mathcal{O}_{r,R}[[\mathbf{z}]]$
$\mathcal{A}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$	‖ the set of x-normalised automorphisms of $\mathcal{O}_R[[\mathbf{z}]]$
$\mathcal{A}_{norm}(\mathbb{C}[[x,\mathbf{z}]])$	‖ the set of x-normalised formal automorphisms
$\mathcal{D}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$	‖ the set of x-normalised derivations of $\mathcal{O}_R[[\mathbf{z}]]$
$>_{gr.lex}$	‖ the graded lexicographic order
$\mathrm{NR}(S)$	‖ the set $\{T \in \mathcal{L}_n; \langle \mu, T\rangle \in \mathbb{Z}_{\geq 1}\}$, where $\mu = (\mu_1, \ldots, \mu_n)$, and μ_i is ‖ eigenvalue of the semisimple derivation S
$\mathrm{PR}(S)$	‖ the set $\{K \in \mathcal{L}_n \setminus \{\mathbf{0}\}; \langle \mu, T\rangle \in \mathbb{Z}_{\leq 0}\}$, where $\mu = (\mu_1, \ldots, \mu_n)$, and ‖ μ_i is eigenvalue of the semisimple derivation S
X_{lin}	‖ the linear part of a derivation X

Chapter 5

$\mathcal{R}\mathrm{S}$	‖ a Riemann surface
$\mathbb{D}$	‖ a poli-disk around the origin in $\mathbb{C}^n$
$\arg z$	‖ the argument of a complex number z
$\mathrm{Im}(z)$	‖ the imaginary part of a complex number z
$\mathrm{Re}(z)$	‖ the real part of a complex number z
$\mathfrak{R}$	‖ the region $\{z \in \mathbb{C}; \pi \leq \arg z \leq \pi + \arg \lambda\}$
$\mathcal{C}$	‖ the discrete positive cone $\{p_1\lambda + p_2\mu; p_1, p_2 \in \mathbb{Z}_{\geq -1}, p_1 + p_2 \geq 0\}$, ‖ where $\lambda, \mu \in \mathbb{C}$

Chapter 1

Regular analytic foliations and their holonomies

In this chapter, we make a quick presentation of the regular foliation theory which will work as north to our present study. In the first section, we formalize some notations about vector fields. The second section exposes some basic facts about regular foliations and their holonomies. A detailed exposition of these topics can be found in [CN77], [Sot11], and [IY08].

A *regular foliation* is an equivalence relation on a manifold where the equivalence classes are connected, injectively immersed submanifolds of the same dimension. A famous example of regular foliation is the foliation of the torus developed by Reeb, see Figure 1.1.

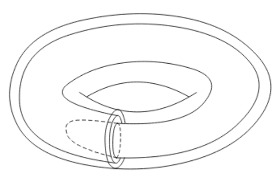


Figure 1.1: Reeb foliation of the torus

As discussed in the introduction, we study analytic foliations that are locally generated by vector fields. The leaves of such foliations correspond locally to the *integral curves* of such vector fields.

The holonomy group of a leaf describes the transverse behaviour of a foliation in the vicinity of such a leaf, and it is an object of primary importance to the present work. We give a detailed definition and show its relation to the analytic equivalence of foliations (Proposition 1.2.13). One of the main motivations of this work was to try to generalize this last proposition to the case of singular foliations.

1.1 Vector fields

Let $\mathcal{M}$ be a complex manifold. The *tangent bundle* of $\mathcal{M}$ is a manifold $\mathcal{TM}$ which assembles all the tangent vectors of $\mathcal{M}$. As a set, it is given by the disjoint union of the tangent spaces of $\mathcal{M}$. That is,

$$\mathcal{TM} = \bigsqcup_{p \in M} \mathcal{T}_p\,\mathcal{M}$$

where $\mathcal{T}_p\,\mathcal{M}$ denotes the *tangent space* to $\mathcal{M}$ at a point p. An element of $\mathcal{TM}$ can be thought of as a pair (p, v), where p is a point in $\mathcal{M}$ and v is a tangent vector to $\mathcal{M}$ at p. Moreover, there is a natural projection $\pi : \mathcal{TM} \to \mathcal{M}$ defined by $\pi(p, v) = p$. This projection maps each element of the tangent space $\mathcal{T}_p\,\mathcal{M}$ to the single point p.

Given a chart $\phi : U \to \mathbb{C}^n$ of $\mathcal{M}$, we can define a chart $\overline{\phi} : \pi^{-1}(U) \to U \times \mathbb{C}^n$ of $\mathcal{TM}$ by $\overline{\phi}(p, v) = (\phi(p), \mathrm{d}(\phi(p))v)$. With this manifold structure in $\mathcal{TM}$, we can define the vector fields $\partial_{\mathrm{z}_1}, \ldots, \partial_{\mathrm{z}_\mathrm{n}} : U \to \mathcal{TM}$ by $\partial_{\mathrm{z}_\mathrm{i}}(\mathbf{z}) = \phi^*(e_i) = \overline{\phi}^{-1}(\phi(\mathbf{z}), e_i)$. In particular, these vector fields form a basis of $\mathcal{T}_p\,\mathcal{M}$ at any $p \in \mathcal{M}$. Consequently, we can write any analytic vector field $X : U \to \mathcal{TM}$ in terms of $\partial_{\mathrm{z}_1}, \ldots \partial_{\mathrm{z}_\mathrm{n}}$, i.e.,

$$X(\mathbf{z}) = \sum_{i=1}^{n} f_i(\mathbf{z})\,\partial_{\mathrm{z}_\mathrm{i}},$$

where $f_i : U \to \mathbb{C}$ is an analytic function for $i = 1, \ldots, n$.

In the literature, vector fields have many different interpretations. In particular, we explore the meaning of derivations and ordinary differential equations. In this section, we introduce the connection with ordinary differential equations. The study of derivations will be done in Section 3.2.

Let U be an open of $\mathcal{M}$, and $X : U \to \mathcal{TM}$ be an analytic vector field of the form $X = \sum_{i=1}^{n} f_i(\mathbf{z})\,\partial_{\mathrm{z}_\mathrm{i}}$. The *ordinary differential equation associated with* X, which is denoted by $\mathbf{z}' = X(\mathbf{z})$, is

$$\begin{aligned} \frac{\partial z_1}{\partial t} &= f_1(z_1(t), \ldots, z_n(t)), \\ &\vdots \\ \frac{\partial z_n}{\partial t} &= f_n(z_1(t), \ldots, z_n(t)). \end{aligned}$$

Given a point $p \in U$, a solution for an equation $\mathbf{z}' = X(\mathbf{z})$ is an analytic map $F_X(\cdot, p) : V \to U$, where $V \subset \mathbb{C}$ is a simply connected open neighborhood of $0 \in \mathbb{C}$, such that, for all $t \in V$,

$$\begin{aligned} \frac{\partial F_X(t,p)}{\partial t} &= X(F_X(t,p)), \\ F_X(0,p) &= p. \end{aligned} \tag{1.1}$$

We call such solutions of *integral curves of* X *with initial value* $z(0) = p$, or simply integral curves of X when the initial value p can be omitted.

Proposition 1.1.1 ([IY08], Theorem 1.1)**.** *Let $U \subset \mathbb{C} \times \mathbb{C}^n$ be an open domain, and $X : U \to \mathbb{C}^n$ be an analytic vector field. Let us consider the equation*

$$\boldsymbol{z}' = X(t, \boldsymbol{z}),$$

and a given point $p = (t_0, p_1, \ldots, p_n) \in U$. Then there exists a sufficiently small polydisk $D^n_\epsilon = \{\|t-t_0\|< \epsilon, \|z_j - p_j\|< \epsilon, j = 1, \ldots, n\} \subset U$, such that a solution with initial value $z(t_0) = p$ exists and is unique in this polydisk.

This solution depends holomorphically on the initial value $p \in \mathbb{C}^n$ and on any additional parameters, provided that X depends holomorphically on these parameters.

Definition 1.1.2 (*Vector fields' local conjugation*)**.** *Let $\mathcal{M}, \mathcal{N}$ be analytic manifolds, and $X_1 : \mathcal{M} \to \mathcal{TM}$ and $X_2 : \mathcal{N} \to \mathcal{TN}$ be analytic vector fields. We say that X_1 is locally analytically conjugated to X_2 at $p \in \mathcal{M}$ if there exist V_1, V_2 opens of $\mathcal{M}, \mathcal{N}$, respectively, and a bianalytic map $\psi : V_1 \to V_2$ such that $p \in V_1$, $\psi(p) \in V_2$, and*

$$\mathrm{d}\,\psi|_{\boldsymbol{z}} X_1(\boldsymbol{z}) = X_2(\psi(\boldsymbol{z})) \tag{1.2}$$

for all $\boldsymbol{z} \in V_1$.

Keeping the notation of the definition above, let $F_{X_1}(t, p)$ be a solution of the ODE associated with X_1 with the initial value problem $z(0) = p$. Taking the derivative with respect to t of $\psi(F_{X_1}(t, p))$, we have

$$\begin{aligned}(\psi(F_{X_1}(t,p)))' &= \mathrm{d}\,\psi|_{F_{X_1}(t,p)} \frac{\partial}{\partial_t} F_{X_1}(t,p) = \mathrm{d}\,\psi|_{F_{X_1}(t,p)} X_1(F_{X_1}(t,p)) \\ &= X_2(\psi(F_{X_1}(t,p))).\end{aligned}$$

Then $\psi(F_{X_1}(t, p))$ is a solution for the problem $\mathbf{z}' = X_2(\mathbf{z})$, with $z(0) = \psi(p)$. Consequently, by Proposition 1.1.1, we have that

$$\psi(F_{X_1}(t, p)) = F_{X_2}(t, \psi(p)) \tag{1.3}$$

in the intersection of the domains. Therefore, conjugated vector fields have correlated integral curves as shown in (1.3).

1.2 Holonomy of a regular foliation

In this section, we define the concepts of regular foliation and holonomy group of a leaf. We show that, under appropriate hypothesis, the holonomy classifies the foliation up to equivalence.

Definition 1.2.1 (*Regular analytic foliation*)**.** *Let $\mathcal{M}$ be a complex manifold of dimension n. A regular analytic foliation $\mathcal{F}$ of $\mathcal{M}$ of complex dimension s is a maximal holomorphic atlas*

$$\{(U_j, \phi_j) | \phi_j : U_j \to \phi_j(U_j) \subset \mathbb{C}^s \times \mathbb{C}^{n-s}\},$$

such that the transition maps

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_j \cap U_i) \to \phi_j(U_j \cap U_i)$$

have the form

$$(x, y) \mapsto (g_{ij}(x, y), h_{ij}(y)), \quad x \in U \subset \mathbb{C}^s, \quad y \in V \subset \mathbb{C}^{n-s},$$

where g_{ij}, h_{ij} *are holomorphic maps.*

We call a *plaque* a set of the form $\phi^{-1}(V \times \{c\})$, where $V \subset \mathbb{C}^s$ is an open set and $c \in \mathbb{C}^{n-s}$, and a *leaf* the maximal connected component of a plaque.

Definition 1.2.2 (*Transverse section*)**.** *Let $\mathcal{M}$ be a complex manifold of dimension n, $\mathcal{F}$ an analytic foliation of dimension s on an open set $U \subset \mathcal{M}$, $\mathcal{P}$ a plaque of $\mathcal{F}$, and $p \in \mathcal{P}$ a point. A parametrized transverse section of $\mathcal{P}$ at p is an analytic map $\tau : (\mathbb{C}^{n-s}, \boldsymbol{0}) \to (U, p)$ transverse to $\mathcal{L}$, i.e.,*

$$T_p U = \mathrm{d}\,\tau(\boldsymbol{0})(T_{\boldsymbol{0}}\mathbb{C}^{n-s}) \oplus T_p\,\mathcal{P}\,.$$

The image of τ is called a transverse section of $\mathcal{P}$ at p.

Consider a chart (U, ϕ) of a foliation $\mathcal{F}$ and a transverse section τ_1 at a point $p_1 = \phi^{-1}(w_0, c)$. Then, we can write

$$\phi \circ \tau_1(w) = (g_{p_1}(w), h_{p_1}(w)),$$

with $h'_{p_1}(0) \neq 0$, consequently, the map h_{p_1} is invertible in a neighborhood of $h_{p_1}(0) = c$.

Let $p_2 = \phi^{-1}(w_1, c)$ be a point in the same plaque $\mathcal{P}$ of p_1, and $\tau_2 = \phi^{-1} \circ (g_{p_2}, h_{p_2})$ be a transverse sections at p_2. Since h_{p_1} and h_{p_2} are invertible at $h_{p_1}(0) = h_{p_2}(0) = c$, we can define an analytic map $\Delta_{\tau_1,\tau_2} : (\tau_1, p_1) \to (\tau_2, p_2)$ given by

$$\Delta_{\tau_1,\tau_2}(w) = \tau_2 \circ h_{p_2}^{-1} \circ h_{p_1} \circ {\tau_1}^{-1}(w). \tag{1.4}$$

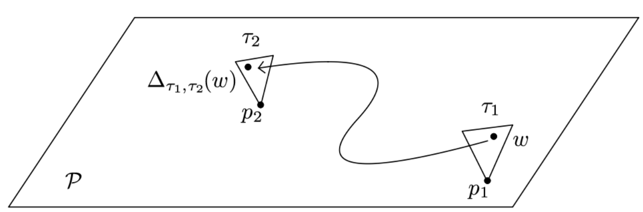


Figure 1.2: The correspondence map Δ_{τ_1,τ_2}.

For the sake of completeness, we are going to enunciate a sequence of results that are well-known. For the proofs of Claims 1.2.3 to 1.2.12, we refer to [Aba09] pages 21-26 or [CN77] page 62-66.

Claim 1.2.3. *The germ of the map Δ_{τ_1,τ_2} defined in (1.4) is independent, up to an analytic change of coordinates, on the parametrized transverse section and the charts of the foliation.*

Definition 1.2.4 (*Correspondence map*)**.** *Let $\mathcal{P}$ be a plaque of an analytic foliation of a complex manifold. Given two points $p_1, p_2 \in \mathcal{P}$, and two transverse sections τ_1, τ_2 at p_1 and p_2 respectively, the map $\Delta_{\tau_1,\tau_2} : (\tau_1, p_1) \to (\tau_2, p_2)$ defined in (1.4) is called the correspondence map between τ_1 and τ_2.*

Consider a leaf $\mathcal{L}$ of a foliation $\mathcal{F}$, two points $p_1, p_2 \in \mathcal{L}$, and a path $\gamma : [0,1] \to \mathcal{L}$ with $\gamma(0) = p_1$ and $\gamma(1) = p_2$. Since the image of γ is compact, we can find a finite cover $\{U_k\}_{k=1,\ldots,n}$ of $\gamma([0,1])$ by charts of $\mathcal{F}$. Consider a partition $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ such that $\gamma([t_j, t_{j+1}]) \subset U_{j+1}$. In addition, at each $\gamma(t_j)$, let τ_j be a transverse section. Hence we can define the following map

$$\Delta_\gamma := \Delta_{\tau_{n-1},\tau_n} \circ \cdots \circ \Delta_{\tau_0,\tau_1} . \tag{1.5}$$

Claim 1.2.5. *The map Δ_γ does not depend on the charts U_k's neither the intermediate points $\gamma(t_k)$*

Definition 1.2.6 (*Holonomy map associated with a curve*)**.** *Let $\mathcal{L}$ be a leaf of an analytic foliation of a complex manifold, and $\gamma : [0,1] \to \mathcal{L}$ be a path in $\mathcal{L}$. Consider two transverse sections τ_1 and τ_2 at $p_1 = \gamma(0)$ and $p_2 = \gamma(1)$, respectively. The map Δ_γ defined in (1.5) computed with respect to τ_1 and τ_2 is called the holonomy map associated with γ.*

Definition 1.2.7 (*The first homotopy group*)**.** *The first homotopy group with base at a point $p \in \mathcal{M}$ is the set of homotopy classes of continuous maps*

$$f : \mathbb{S}^1 \to \mathcal{M},$$

where $\mathbb{S}^1 := \{z \in \mathbb{C}; \|z\| = 1\}$ and $f(1) = p$. We denote by $\pi_1(\mathcal{L}, p)$ the homotopy group at a point $p \in \mathcal{M}$.

Claim 1.2.8. *Let $\mathcal{L}$ be a leaf of an analytic foliation of a complex manifold, $\gamma_1, \gamma_2 : [0,1] \to \mathcal{L}$ be paths in $\mathcal{L}$, where $\gamma_1(0) = \gamma_2(0) = p_1$ and $\gamma_1(1) = \gamma_2(1) = p_2$, and τ_1 and τ_2 be two transverse sections at p_1 and p_2, respectively. If γ_1 and γ_2 are homotopic equivalent, then $\Delta_{\gamma_1} = \Delta_{\gamma_2}$ as elements of $\pi_1(\mathcal{L}, p)$.*

An *antihomomorphism* between two groups $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is a homomorphism $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2^{\text{op}}$, where $\mathfrak{g}_2^{\text{op}}$ is equals $\mathfrak{g}_2$ as a set, but has its multiplication reversed to that defined on $\mathfrak{g}_2$. Denoting the multiplication on $\mathfrak{g}_2$ by $\cdot$, the multiplication on $\mathfrak{g}_2^{\text{op}}$, denoted by $*$, is defined by $x * y := y \cdot x$. The group $\mathfrak{g}_2^{\text{op}}$ is called the *opposite group* to $\mathfrak{g}_2$.

Claim 1.2.9 (*The map Δ_γ is group antihomomorphism*)**.** *Let $\mathcal{L}$ be a leaf of an analytic foliation of a complex manifold, and p be a point in $\mathcal{L}$. The map which associates to $[\gamma] \in \pi_1(\mathcal{L}, p)$ its holonomy map Δ_γ is an antihomomorphism of groups.*

Definition 1.2.10 (*Holonomy group at a point*)**.** *The image* $\mathrm{Hol}(\mathcal{L}, p)$ *of the map defined in the Claim 1.2.9 is the holonomy group of* $\mathcal{L}$ *at* p*.*

Definition 1.2.11 (*Conjugated holonomy groups*)**.** *We say that two holonomy groups* $\mathrm{Hol}(\mathcal{L}_1, p_1)$ *and* $\mathrm{Hol}(\mathcal{L}_2, p_2)$ *computed at respective transverse sections* τ_1, τ_2 *at* p_1, p_2 *are analytically conjugated if there exists a germ of bianalytic map* $\varphi : (\tau_1, p_1) \to (\tau_2, p_2)$ *such that*

$$\varphi \circ \mathrm{Hol}(\mathcal{L}_1, p_1) \circ \varphi^{-1} = \mathrm{Hol}(\mathcal{L}_2, p_2).$$

Claim 1.2.12. *Let* $\mathcal{L}$ *be a leaf of an analytic foliation of a complex manifold. Then for any* $p_1, p_2 \in \mathcal{L}$ *the holonomy groups* $\mathrm{Hol}(\mathcal{L}, p_1)$ *and* $\mathrm{Hol}(\mathcal{L}, p_2)$ *are analytic conjugated.*

In other words, this claim asserts that the holonomy group does not depend on the base point. Then, we can define the *holonomy group for a leaf* $\mathrm{Hol}(\mathcal{L})$ as being the holonomy group at any point in the leaf.

The following well-known result (see, for instance [CN85], page 67) establishes the relation between the conjugation of holonomies and the local equivalence of foliations of class C^s with $1 \leq s \leq \infty$.

Proposition 1.2.13 ([CN85], Theorem 2, Chapter IV)**.** *Let* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *be compact* C^s*-diffeomorphic leaves of foliations* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *respectively. The holonomy of* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *are* C^s *conjugated if only there exist neighborhoods* $V_1 \supset \mathcal{L}_1$ *and* $V_2 \supset \mathcal{L}_1$*, and a* C^s *diffeomorphism* $\Phi : V_1 \to V_2$, $\Phi(\mathcal{L}_1) = \mathcal{L}_2$*, taking leaves of* $\mathcal{F}_1|_{V_1}$ *to leaves of* $\mathcal{F}_2|_{V_2}$*. In this case, we say* $\mathcal{F}_1$ *and* $\mathcal{F}_2$ *are locally equivalent on* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *and* Φ *is a local equivalence.*

Remark 1.2.14. *Notice that the proof of Proposition 1.2.13, as presented in [CN85], is not valid for holomorphic foliations. In fact, their proof is strongly based on the existence of a local transverse fibration to a leaf, which does not necessarily exists in the complex analytic setting.*

Chapter 2

Endomorphisms, automorphisms, and derivations

Analytic vector fields can be seen as derivations in the ring of analytic functions. More generally, formal vector fields also have this interpretation, but in the ring of formal series. In the first section, we define precisely the concept of formal rings with coefficients in some specific commutative rings. In the second section, we define automorphisms and derivations in the rings described in the first section. We will see that a change of coordinates can be seen as the action of an automorphism on the corresponding ring of functions.

2.1 The formal rings

Given a commutative ring $\mathcal{R}$, our goal is to construct the ring of formal power series of n indeterminates $z_1, \ldots, z_n$ with coefficients in $\mathcal{R}$. It will allow us to treat formal change of coordinates and formal vector fields in the algebraic sense of endomorphisms in a ring. The construction is based on the notion of inverse limit.

Let $\mathcal{R}[\mathbf{z}]$ be the ring of polynomials in the variables $z_1, \ldots, z_n$ with coefficients in $\mathcal{R}$. Consider the maximal ideal $\mathfrak{m} := \langle z_1, \ldots, z_n \rangle \subset \mathcal{R}[\mathbf{z}]$. The *k-jet space* is the quotient $J^{(k)} = \mathcal{R}[\mathbf{z}]/\mathfrak{m}^{k+1}$, where $\mathfrak{m}^{k+1}$ is the ideal generated by the monomials of degree k+1.

Consider $(J^{(i)}, (\pi_{ji}))_{j \le i \in \mathbb{Z}_{\ge 0}}$ the inverse system indexed by $\mathbb{Z}_{\ge 0}$, where for each $j \le i$, $\pi_{ji} : J^{(i)} \to J^{(j)}$ is the linear map with kernel $\mathfrak{m}^{j+1}$ called *bonding map*. We define

$$\mathcal{R}[[\mathbf{z}]] := \varprojlim_{j \in \mathbb{N}} J^{(i)} = \left\{ f \in \prod_{i \in \mathbb{N}} J^{(i)}; f_j = \pi_{ji}(f_i) \text{ for all } j \le i \in \mathbb{N} \right\}.$$

An element $f \in \mathcal{R}[[\mathbf{z}]]$ is called a *formal series with coefficients in* $\mathcal{R}$, and we write $f(\mathbf{z}) = \sum_{|K|=0}^{\infty} c_K \mathbf{z}^K$, where $K \in \mathbb{Z}^n_{\geq 0}$, $\mathbf{z}^K = z^{k_1} \dots z^{k_n}$, $|K| = k_1 + \dots + k_n$, and each coefficient $c_K \in \mathcal{R}$. We still denote by $\mathfrak{m}$ the maximal ideal $\langle z_1, \dots, z_n \rangle \subset \mathcal{R}[[\mathbf{z}]]$.

We can identify the ring of polynomials of degree less or equal k with the k-jet space $J^{(k)} \subset \mathcal{R}[[\mathbf{z}]]$. For any $k \in \mathbb{Z}_{\geq 1}$, we can define a *truncation map* $\pi_k : \mathcal{R}[[\mathbf{z}]] \to J^{(k)}$ by $\pi_k(f) := f \mod \mathfrak{m}^{k+1}$. The image of $f \in \mathcal{R}[[\mathbf{z}]]$ by π_k is called by the k^{th}*-truncation* of f.

In this text, we work with some specific commutative rings as coefficients. More specifically, we are going to consider the following rings.

1. The ring $\mathbb{C}[[\mathbf{z}]]$ of *formal power series* $\sum_{|K|=0}^{\infty} a_K \mathbf{z}^K$, where $a_K \in \mathbb{C}$.

2. The ring $\mathcal{O}_R[[\mathbf{z}]]$ of D_R*-transversely formal series*. A D_R-transversely formal series is a power series in the variables $z_1, \dots, z_n$ and coefficients in the ring $\mathcal{O}_R$ of analytic functions in the disk $D_R = \{x \in \mathbb{C} : \|x\| < R\}$, where $R > 0$. Explicitly, an element $f \in \mathcal{O}_R[[\mathbf{z}]]$ can be written as

$$f(x, \mathbf{z}) = \sum_{|K|=0}^{\infty} f_K(x) \mathbf{z}^K, \tag{2.1}$$

   where each coefficient $f_K(x)$ is convergent in the disk $D_R \subset \mathbb{C}$.

3. The ring $\mathcal{O}_{r,R}[[\mathbf{z}]]$ of $D_{r,R}$*-transversely formal series*. A $D_{r,R}$-transversely formal series is a power series in the variables $z_1, \dots, z_n$ and coefficients in the ring $\mathcal{O}_{r,R}$ of analytic functions in the annulus $D_{r,R} = \{x \in \mathbb{C} : r < \|x\| < R\}$, where $r, R > 0$. Explicitly, an element $f \in \mathcal{O}_{r,R}[[\mathbf{z}]]$ can be written as (2.1) where each coefficient $f_K(x)$ is convergent in the disk $D_{r,R} \subset \mathbb{C}$.

For the three cases, we still denote by $\mathfrak{m}$ the respective maximal ideal $\langle z_1, \dots, z_n \rangle$.

2.2 The subrings $\mathcal{O}_R\{\mathbf{z}\}$ and $\mathcal{O}_{r,R}\{\mathbf{z}\}$ of convergent series

We now consider the subring of convergent elements $\mathcal{O}_R\{\mathbf{z}\}$ and $\mathcal{O}_{r,R}\{\mathbf{z}\}$ in the above formal rings.

Definition 2.2.1 (*The ring of D_R-transversely convergent series*)**.** *We say that $f \in \mathcal{O}_R[[\mathbf{z}]]$, written as in (2.1), is convergent if for each R' with $0 < R' < R$, there exist constants $C, M \in \mathbb{R}$ such that $\sup_{D_{R'}} \|f_K\| \leq CM^{|K|}$. We denote by $\mathcal{O}_R\{\mathbf{z}\} \subset \mathcal{O}_R[[\mathbf{z}]]$ the ring of D_R-transversely convergent series.*

Definition 2.2.2 (*The ring of $D_{r,R}$-transversely convergent series*)**.** *We say that $f \in \mathcal{O}_{r,R}[[\mathbf{z}]]$, written as in (2.1), is convergent if for each r', R' with $0 < r < r' < R' < R$, there exist constants $C, M \in \mathbb{R}$ such that $\sup_{D_{r',R'}} \|f_K\| \leq CM^{|K|}$. We denote by $\mathcal{O}_{r,R}\{\mathbf{z}\} \subset \mathcal{O}_{r,R}[[\mathbf{z}]]$ the ring of $D_{r,R}$-transversely convergent series.*

We observe that the diagram of inclusions in Figure 2.1 follows directly from the definitions. In particular, the next lemma establishes that the ring in the last row is the intersection of the two in the middle line.

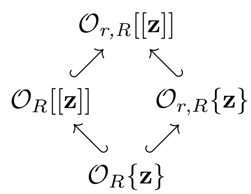


Figure 2.1: Diagram of inclusions

Lemma 2.2.3. $\mathcal{O}_{r,R}\{\mathbf{z}\} \cap \mathcal{O}_R[[\mathbf{z}]] = \mathcal{O}_R\{\mathbf{z}\}$.

The proof is based on the next claim.

Claim 2.2.4. *Let f be an analytic function on D_R. Then $\sup_{D_{r,R}} \|f\| = \sup_{D_R} \|f\|$, for any $0 < r < R$.*

Proof. The Maximum Modulus Principle for analytic functions guarantees that the $\sup_A \|f\|$ is attained on the boundary of A. As $\partial D_{r,R} \subset \overline{D_R}$ and $\partial D_R \subset \partial D_{r,R}$, it follows that $\sup_{D_{r,R}} \|f\| = \sup_{D_R} \|f\|$. □

proof of Lemma 2.2.3. By the above claim, as $f \in \mathcal{O}_{r,R}\{\mathbf{z}\}$, each coefficient f_K satisfies

$$\sup_{D_{R'}} \|f_K\| = \sup_{D_{r',R'}} \|f_K\| \leq CM^K,$$

for any $0 < r' < R' < R$. Consequently, $f \in \mathcal{O}_R\{\mathbf{z}\}$. The reciprocal is immediate. □

2.3 Endomorphisms

Let $\mathcal{R}$ be one of the rings $\mathbb{C}$, $\mathcal{O}_R$ or $\mathcal{O}_{r,R}$. A $\mathbb{C}$-linear endomorphism of $\mathcal{R}[[\mathbf{z}]]$ is defined by a sequence indexed by $\mathbb{Z}_{\geq 0}$ of $\mathbb{C}$-linear maps

$$\Phi^{(i)} \in \operatorname{Hom}_{\mathbb{C}}(\mathcal{R}[[\mathbf{z}]], J^{(i)}),$$

such that, for each $j < i$, $\pi_{ji} \circ \Phi^{(i)} = \Phi^{(j)}$, where π_{ji} is a bonding map, and

$$\Phi(f) = \prod_{i \in \mathbb{Z}_{\geq 0}} \Phi^{(i)}(f),$$

We denote by $\operatorname{End}_{\mathbb{C}}(\mathcal{R}[[\mathbf{z}]])$ the set of such endomorphisms, and we say that $\Phi^{(i)}$ is the i^{th}-truncation of Φ.

We say that a $\mathbb{C}$-linear endomorphism of $\mathcal{R}[[\mathbf{z}]]$ is an *automorphism* if it satisfies the following additional properties:

1. $\Phi(\mathfrak{m}) \subset \mathfrak{m}$,
2. $\Phi(\alpha f) = \alpha\Phi(f)$,
3. $\Phi(fg) = \Phi(f)\Phi(g)$,

where $f, g \in \mathcal{R}[[\mathbf{z}]]$, fg denotes the product of f and g. We denote by $\mathcal{A}(\mathcal{R}[[\mathbf{z}]])$ the group of such automorphisms.

We observe that $\mathcal{A}(\mathbb{C}[[\mathbf{z}]])$ can be interpreted as the group of formal changes of coordinates. Indeed, any element $\Phi \in \mathcal{A}(\mathbb{C}[[\mathbf{z}]])$ is defined by its components $\Phi_i(z) := \Phi(z_i)$, $1 \leq i \leq n$, where they satisfy

$$\mathrm{Jac}(\Phi) = \det \begin{pmatrix} \frac{\mathrm{d}\,\Phi_1}{\mathrm{d}\,z_1} & \cdots & \frac{\mathrm{d}\,\Phi_1}{\mathrm{d}\,z_n} \\ \vdots & \ddots & \vdots \\ \frac{\mathrm{d}\,\Phi_n}{\mathrm{d}\,z_1} & \cdots & \frac{\mathrm{d}\,\Phi_n}{\mathrm{d}\,z_n} \end{pmatrix} (\mathbf{0}) \neq 0.$$

By analogy, an element of $\mathcal{A}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ (resp. $\mathcal{A}(\mathcal{O}_R[[\mathbf{z}]])$) will be called a *$D_{r,R}$-transversely formal change of coordinates* (resp. a *D_R-transversely formal change of coordinates*).

Similarly, a vector field with components in $\mathcal{R}[[\mathbf{z}]]$ can be seen as a derivation on this ring. That is, an endomorphism $X \in \mathrm{End}_{\mathbb{C}}(\mathcal{R}[[\mathbf{z}]])$, satisfying the Leibniz's rule

$$X(f.g) = X(f).g + X(g).f \quad \forall f, g \in \mathcal{R}[[\mathbf{z}]].$$

We denote by $\mathcal{D}(\mathcal{R}[[\mathbf{z}]])$ the $\mathcal{R}[[\mathbf{z}]]$-module of such derivations that also satisfy the condition $X(\mathfrak{m}) \subset \mathfrak{m}$. Note that $\mathcal{D}(\mathcal{R}[[\mathbf{z}]])$ has a Lie Algebra structure with the Lie Bracket given by

$$[X, Y] = X \circ Y - Y \circ X.$$

In particular, an element of $\mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ (resp. $\mathcal{D}(\mathcal{O}_R[[\mathbf{z}]])$) will be called *$D_{r,R}$-transversely formal derivation* (resp *D_R-transversely formal derivation*).

Chapter 3

Singular foliations: holonomy and normal forms

The classification of singular holomorphic foliations is a classical subject which dates to the pioneering works of Dulac and Poincaré (see, e.g. [Dul23]). In this chapter, we recall some basic definitions that are used in the following sections such as singular foliations, separatrices, holonomy, etc. We will also review some basic results of normal forms of vector fields and the basic construction of the equivalence of foliations via the path lifting method. We refer to [RR11] for a detail exposition. From now on, we will use the word singular foliation as a synonym for singular foliations of dimension one, since this is the only case that will be considered in the sequel.

3.1 Singular foliations and holonomy

Let $\mathcal{M}$ be a complex holomorphic manifold of dimension $n \geq 1$.

Definition 3.1.1 (*Singular analytic foliation*)**.** *A (one-dimensional) singular holomorphic foliation $\mathcal{F}$ on $\mathcal{M}$ is defined by the following data:*

1. *An open covering $\bigcup_{i \in A} U_i$ of $\mathcal{M}$,*
2. *An analytic vector X_i on each open set U_i,*

such that for each intersection $U_i \cap U_j \neq \varnothing$ there exists a non-vanishing analytic function $h_{ij} : U_i \cap U_j \to \mathbb{C}^*$ such that:

$$X_i = h_{ij} X_j.$$

The vector fields X_i in the definition above are called *the local generators* of the foliation. The *singular set* Σ of a singular foliation $\mathcal{F}$ is locally defined by the singular set of its local generators, i.e., $\Sigma = \{p \in \mathcal{M}; X_i(p) = 0$, where X_i is a local generator of $\mathcal{F}\}$. The following result shows that we can always assume that Σ has codimension greater or equal than two.

Proposition 3.1.2 ([Aba09], Theorem 1.2.9)**.** *Let $\mathcal{F}_1$ be singular foliation with singular set Σ_1. Then, there exists a singular foliation $\mathcal{F}_2$, with singular set $\Sigma_2 \subset \Sigma_1$ of codimension at least 2, whose leaves coincide with those of $\mathcal{F}_1$ outside Σ_1.*

In this work, we are interested germs of singular foliations.

Definition 3.1.3 (*A germ of foliation*)**.** *A germ of foliation at $p \in \mathcal{M}$ is a foliation $\mathcal{F}$ defined by the solution curves of germ of analytic vector field X, i.e. a derivation of the local ring $\mathcal{O}_{\mathcal{M},p}$ of germs of holomorphic functions at p.*

As above, we will say that X is a local generator of $\mathcal{F}$. Notice that if $u \in \mathcal{O}_{\mathcal{M},p}$ is a unit, then X and uX are both local generators of $\mathcal{F}$.

Definition 3.1.4 (*Separatrix*)**.** *A separatrix of $\mathcal{F}$ is a germ of irreducible analytic curve $\mathcal{S}$ whose defining ideal $I = I_S$ satisfies*

$$X(I) \subset I,$$

where X is any local generator of X (acting on each $f \in I$ as a derivation).

Let us define the holonomy of $\mathcal{F}$ along a separatrix $\mathcal{S}$. To simplify the exposition, and since this is the only case that will be needed, we will consider the case where $\mathcal{S}$ is a smooth curve and that $\mathcal{S} \cap \Sigma$ has an isolated point at p (where we recall that Σ is the singular set of $\mathcal{F}$). Therefore, we can choose a local coordinate chart $\mathbf{z} = (z_1, \dots, z_n)$ at p, such that $\mathcal{S} = \{z_2 = \dots = z_n = 0\}$. We can further assume that the image of this chart in $\mathbb{C}^n$ contains the closed disk $\{(z_1, 0, \dots, 0) : \|z_1\| \leq R\}$. Moreover, we can choose a local generator X of $\mathcal{F}$ which has the form

$$X = \sum f_i(z)\,\partial_{\mathrm{z_i}},$$

where f_1 restricted to $\mathcal{S}$ vanishes only at the origin.

Let $p = (p_1, 0, \dots, 0) \in \mathcal{S}$ such that $\|p_1\| \leq R$. Consider the closed curve $\gamma : [0,1] \to \mathcal{S}$ given by $\gamma(t) = pe^{2\pi i t}$. The differential equation associated with X restricted to γ has the form

$$\begin{aligned} \frac{\partial z_2}{\partial t} &= 2i\pi p e^{2i\pi t} \frac{f_2}{f_1}(\gamma(t), z_2, \dots, z_n), \qquad (3.1) \\ &\vdots \\ \frac{\partial z_n}{\partial t} &= 2i\pi p e^{2i\pi t} \frac{f_n}{f_1}(\gamma(t), z_2, \dots, z_n). \end{aligned}$$

In this setup, the (germ of) holonomy is the map $w \mapsto \Delta(w)$ from the transverse section $\tau = \{z_1 = p\}$ into itself obtained by integrating the above differential equation from $t = 0$ to $t = 1$, with initial condition $(z_2, \dots, z_n) = w$.

As in the case of regular foliations, it can be proved that if we choose a different base point or a different transverse section, then the germ of holonomy is the same up to a holomorphic conjugation.

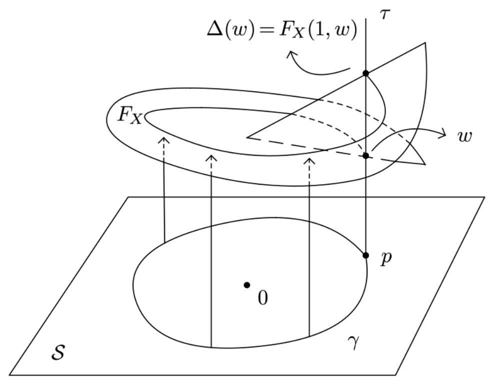


Figure 3.1: Holonomy of a separatrix

3.2 Normal forms for vector fields

For any integer $k \geq 1$, a derivation $X \in \mathcal{D}(\mathbb{C}[[\mathbf{z}]])$ induces a derivation $X^{(k)}$ in the algebra of k-jet spaces of $\mathbb{C}[[\mathbf{z}]]$. In particular, these derivations $X^{(k)}$ admit Jordan decompositions

$$X^{(k)} = X_s^{(k)} + X_n^{(k)},$$

where $[X_s^{(k)}, X_n^{(k)}] = 0$, and $X_s^{(k)}$ and $X_n^{(k)}$ are respectively semisimple and nilpotent parts of $X^{(k)}$. Moreover, these decompositions are compatible by truncation. Therefore, when $k \to \infty$, we obtain the Jordan's decomposition

$$X = X_s + X_n,$$

where $X_s, X_n \in \mathcal{D}(\mathbb{C}[[\mathbf{z}]])$ and $[X_s, X_n] = 0$. We say that X is semisimple (resp. nilpotent) if $X = X_s$ (resp. $X = X_n$).

Now, let $\Phi \in \mathcal{A}(\mathbb{C}[[\mathbf{z}]])$ and $X \in \mathcal{D}(\mathbb{C}[[\mathbf{z}]])$. The conjugation of X by Φ is defined by

$$\Phi^*(X) = \Phi^{-1} \circ X \circ \Phi.$$

In particular, as the Jordan's decomposition of X is the limit of Jordan's decompositions of $X^{(k)}$, we have that $\Phi^*(X_s) + \Phi^*(X_n)$ is the Jordan's decomposition of $\Phi^*(X)$.

Proposition 3.2.1 ([Mar81b], Proposition 1)**.** *Any semisimple derivation $S \in \mathcal{D}(\mathbb{C}[[\mathbf{z}]])$ is conjugated by a formal automorphism to a 'diagonal' derivation*

$$L(\lambda) = \sum_{i=1}^{n} \lambda_i z_i \, \partial_{\mathrm{z_i}}, \tag{3.2}$$

where the numbers $\lambda_i \in \mathbb{C}$ are the eigenvalues S.

Definition 3.2.2 (*Normal form*)**.** *We call a normal form any derivation $X \in \mathcal{D}(\mathbb{C}[[\mathbf{z}]])$ whose semisimple part X_s is a diagonal derivation.*

By Proposition 3.2.1, any derivation is formally conjugated to a normal form. However, note that normal forms are not uniquely defined. Any automorphism that takes X_s to itself generates a possibly new normal form.

For the next lemma, we recall that $\mathcal{L}_n$ is the set of n-uples $K \in \{(\mathbb{N}^n - e_1) \cup \cdots \cup (\mathbb{N}^n - e_n)\}$, such that $|K| \geq 0$.

Lemma 3.2.3 ([Mar81b], Subsections 1.3-1.4)**.** *Let $S \in \mathcal{D}(\mathbb{C}[[\mathbf{z}]])$ be a semisimple derivation with $\lambda_1, \ldots, \lambda_n$ as eigenvalues. Then, the eigenvalues of the brackets operator* $\mathrm{ad}_S := [S, \cdot]$ *on $\mathcal{D}(\mathbb{C}[[\mathbf{z}]])$ are the complex numbers $\alpha_Q = \langle \lambda, Q \rangle$, where $Q = (q_1, \ldots, q_n) \in \mathcal{L}_n$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$.*

Now, let $S \in \mathcal{D}(\mathbb{C}[[\mathbf{z}]])$ be a semisimple derivation. Consider for $k \geq 0$

$$\omega_k = \min \left\{ \|\alpha_Q\|;\ \|Q\| \leq 2^{k+1}, \alpha_Q \neq 0 \right\},$$

where α_Q is an eigenvalues of the map ad_S. We say that a derivation X with semisimple part S satisfies the condition of Brjuno if the series

$$\sum_{k=0}^{\infty} \frac{\log \frac{1}{\omega_k}}{2^k}$$

converges. This condition means that the eigenvalues of the operator $[S, \cdot]$ do not decrease too fast when the order increases.

Proposition 3.2.4 ([Mar81b], page 8)**.** *Let $X \in \mathcal{D}(\mathbb{C}[[\mathbf{z}]])$ be a germ of analytic derivation which satisfies the condition of Brjuno and has $\lambda_1, \ldots, \lambda_n$ as eigenvalues. If X is formally conjugated to a semisimple derivation, then it is analytically conjugated to $L(\lambda) = \sum_{i=1}^{n} \lambda_i z_i \, \partial_{\mathrm{z_i}}$.*

Definition 3.2.5 (*Resonant monomial derivation*)**.** *Let $S = \sum_{i=1}^{n} \lambda_i z_i \, \partial_{\mathrm{z_i}}$ be a diagonal derivation. We say that a derivation of the form*

$$F = z^Q \, \partial_{\mathrm{z_i}},$$

with $Q \in \mathcal{L}_n$, is a resonant monomial derivation for S if F lies in the kernel of the ad_S operator. In other words, if we note $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, then $\langle \lambda, Q \rangle = 0$.

It follows from the commuting relation $[X_s, X_n] = 0$ that if X is a normal form (i.e. X_s is diagonal) then X_n is a formal sum of resonant monomial derivations for X_s.

Consider a derivation $X \in \mathcal{D}(\mathbb{C}[[\mathbf{z}]])$. We denote by $\mathcal{K}(\lambda) \subset \mathbb{C}$ the convex hull of the set $\{\lambda_1, \ldots, \lambda_n\}$ of the eigenvalues X. The next theorem is one of the most important in the normal theory.

Theorem 3.2.6 (Poincare's normal form theorem, [IY08], Section 5.B Theorem 5)**.** *Let $X \in \mathcal{D}(\mathbb{C}[[\mathbf{z}]])$ be an analytic derivation. If the origin $0 \in \mathbb{C}$ does not belong to $K(\lambda)$, then X is analytically conjugated to its normal form.*

The proof is based on the following two facts:

1. The ω_k are limited by a positive constant.

2. The normal forms of X are necessarily polynomials.

3.3 Path lifting method

In this section, we show how the path lifting technique can be used to construct an automorphism which gives an equivalence between germs of analytic vector fields.

Definition 3.3.1 (*Orbitally equivalence of vector fields*)**.** *We say that two germs of singular analytic vector fields X, Y are orbitally analytically equivalent if there exists an analytic germ of automorphism Φ and a unit u such that*

$$\Phi X \Phi^{-1} = uY.$$

As an illustration, we are going to give a brief idea of the proof presented in [MM80a] for the following result.

Theorem 3.3.2. *Let X be a germ of singular analytic vector field in $0 \in \mathbb{C}^2$ with eigenvalues 1 and $\lambda \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$. Then, X is orbitally analytically equivalent to its linear part if and only if the holonomy of a separatrix of the foliation $\mathcal{F}$ generated by X is linearizable.*

The proof is based on the following preliminary result on the existence of local separatrices.

Lemma 3.3.3 ([MM80a], Proposition, Appendix II)**.** *Given a vector field X with eigenvalues λ_1, λ_2 where the quotients $\lambda_1/\lambda_2, \lambda_2/\lambda_1 \notin \mathbb{Z}_{>1}$. Then X is orbitally analytically equivalent to*

$$Z = \lambda_1 z_1 (1 + f(\boldsymbol{z}))\, \partial_{z_1} + \lambda_2 z_2 (1 + g(\boldsymbol{z}))\, \partial_{z_2}, \tag{3.3}$$

where $f(\boldsymbol{0}) = g(\boldsymbol{0}) = 0$. In particular, the curves $\{z_1 = 0\}$ and $\{z_2 = 0\}$ are separatrices of X.

Idea of the proof of Theorem 3.3.2 in [MM80a]. By the hypothesis $\lambda \notin \mathbb{R}_{\geq} 0$, using Lemma 3.3.3, we can assume up to division by a unit, that X is already in the form

$$X = z_1\, \partial_{z_1} + \lambda z_2 (1 + g(z))\, \partial_{z_2},$$

where $g \in \mathfrak{m}$. We denote by $\mathcal{F}$ the foliation generated by X. Let $Y = z_1\, \partial_{z_1} + \lambda z_2\, \partial_{z_2}$, and $\mathcal{G}$ be the foliation generated by Y. Up to a further analytic change of coordinates, we can suppose that $g(z)$ is divisible by z_1 and X is defined in a neighborhood of the disk $\{\|z_1\| \leq 1\} \times \{0\}$. The idea of the proof is to define an analytic diffeomorphism $G : V \to V'$, of the form $G(z_1, z_2) = (z_1, g(z_1, z_2))$ that satisfies the following

1. There exists open neighborhoods U, U' of $\mathbf{0} \in \mathbb{C}^2$ such that $V = U - \{z_1 = 0\}$ and $V' = U' - \{z_1 = 0\}$,
2. The function g is bounded in V,
3. The map G sends leaves of $\mathcal{F}$ to leaves of $\mathcal{G}$ restrict to U and U', respectively.

The construction of G consists in lifting some specific paths contained in the separatrix $\mathcal{S} = \{z_2 = 0\}$ through the leaves of the foliations.

Let $K \subset S$ be the punctured close disk $0 < |z_1| \leq 1$. For each $z_1 \in K$, let $\alpha_{z_1} : [0, -\log\|z_1\|] \to K$ and $\beta_{z_1} : [0, \tilde{t}] \to K$ be the curves such that $\alpha_{z_1}(t) = (z_1 e^t, 0)$, $\beta_{z_1}(t) = \big(z_1/\|z_1\| e^{-2\pi i t}, 0\big)$, and $\beta_{z_1}(\tilde{t}) = (1, 0)$.

Denote by $\mathrm{L}_{\mathcal{F}}^{\gamma}$ and $\mathrm{L}_{\mathcal{G}}^{\gamma}$ the respective lifts of $\gamma \subset K$ to the leaves of $\mathcal{F}$ and $\mathcal{G}$, and by φ the map that conjugates the respective holonomies. The map G is given by

$$G(z_1, z_2) := L_{\mathcal{G}}^{\alpha_{z_1}^{-1}} \circ L_{\mathcal{G}}^{\beta_{z_1}^{-1}} \circ \varphi \circ L_{\mathcal{F}}^{\beta_{z_1}} \circ L_{\mathcal{F}}^{\alpha_{z_1}}(z_1, z_2).$$

By estimating the growth of solution curves from the expression of X given above, one can prove that the diffeomorphism G satisfies the properties (1) and (2) stated above. We refer to [MM80a] for details. $\square$

We now discuss some results generalizing Theorem 3.3.2 to higher dimensions. In the article [EI84], Elizarov and Il'yashenko showed two results, one in dimension 3 and the other in arbitrary dimension with more restricted hypotheses. Later in 2006, Reis simplified the original proof presented in [EI84] for the result in arbitrary dimensions.

Both proofs use refined versions of the path lifting method. One of the delicate points is to guarantee that the equivalence map G constructed by path lifting satisfies the properties (1) and (2) stated in the above proof.

Using the notation of [EI84], let $\mathcal{V}_n$ denote the space of germs analytic vector fields at $\mathbf{0} \in \mathbb{C}^n$ which are singular at the origin and whose linear parts are nondegenerate with simple eigenvalues.

Definition 3.3.4 (*Strict Siegel type*)**.** *We say that a set of complex numbers $\lambda = \{\lambda_1, \ldots, \lambda_n\}$ is of strict Siegel type if the convex hull $\mathcal{K}(\lambda)$ contains a neighborhood of zero.*

Theorem 3.3.5 ([EI84], Theorem 2)**.** *Two germs in $\mathcal{V}_3$ with the same linear part, and the set of eigenvalues being of strict Siegel type are orbitally analytically equivalent if and only if one can choose a local separatrix of each germ such that these separatrices are tangent at the origin and their holonomy maps are analytically conjugated.*

For dimension $n \geq 4$, some stronger restrictions are needed. As stated in the introduction, the authors considered germs of singular analytic vector fields X in $(\mathbb{C}^n, \mathbf{0})$, with $\lambda_1, \ldots, \lambda_n$ as the eigenvalues of the linear part of X, verifying:

1. The origin of $\mathbb{C}^n$ is an isolated singularity of X.

2. X is of Siegel type (i.e, the convex hull of the eigenvalues of its linear part contains the origin in $\mathbb{C}$).

3. There exists a straight line through the origin of $\mathbb{C}$ separating λ_1 from the others eigenvalues in the complex plane.

4. Up to a change of coordinates, $X = \sum_{i=1}^{n} \lambda_i z_i(1 + f_i(z))\partial_{z_i}$, where $z = (z_1, \ldots, z_n)$, and f_i is a germ of analytic function such that $f_i(0) = 0$ for all i.

In [EI84], it is proved the following.

Theorem 3.3.6 ([EI84], Theorem 3)**.** *Let X and Y be two germ vector fields, verifying* (1), (2), (3) *and* (4) *above. Denote by Δ_X and Δ_Y the holonomies of X and Y relatively to the separatrices of X and Y tangent to the eigenspace associated with the first eigenvalue, respectively. Then if Δ_X and Δ_Y are analytically conjugated, X and Y are orbitally analytically equivalent.*

Chapter 4

The exponential map and the Lie bracket

The notion of $D_{r,R}$-transversely formal *nilpotent* derivations and their exponential are crucial to this work. More precisely, the *logarithm*, the inverse map of the exponential map, of a *tangent to the identity* change of coordinates is a *k-flat* nilpotent derivation. With this identification, we can prove the following useful formula

$$(\exp X)^{*}Y = X + [X,Y] + \frac{1}{2}[X,[X,Y]] + \dots. \tag{4.1}$$

In the first section, the notion of *nilpotent* derivation in $\mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ is introduced. In the sequel, we explicit the one-to-one correspondence between the nilpotent derivations and diffeomorphisms that is given by the *exponential map*.

In Section 4.2, the map Ad and the *exponential decomposition* of any *x-normalized* $D_{r,R}$-transversely formal change of coordinates are defined. In particular, this last one allows us to extend, in some sense, the Formula (4.1). However, the more important concept introduced in this chapter is of $D_{r,R}$-transversely formal *symmetries.* Indeed, the technique developed by Arame Diaw [Dia19], which is explored here, is based on the symmetries of the local generators of the foliations.

In Section 4.3, we explicitly calculate a normal form for $D_{r,R}$-transversely formal derivations, and we characterize x-normalized $D_{r,R}$-transversely formal derivations that commute with of an x-normalized D_R-transversely formal nilpotent derivation. Additionally, we conclude in Proposition 4.3.3 that if two x-normalized derivations are x-normalized formal conjugated (the formal change of coordinates is x-normalized), then they are x-normalized D_R-transversely formal conjugated.

4.1 The exponential map

Definition 4.1.1 (*Summable sequence*)**.** *A sequence $\{\Phi_k\}_{k\in\mathbb{N}}$ of endomorphism in $\mathcal{O}_{r,R}[[\mathbf{z}]]$ is called summable if for each $j \in \mathbb{N}$ there exists a natural number $K = K(j)$ such that the j^{th}-truncation of Φ_k is zero for all $k \geq K$.*

Lemma 4.1.2. *A summable sequence of endomorphism $\{\Phi_k\}_{k\in\mathbb{N}}$ in $\mathcal{O}_{r,R}[[\mathbf{z}]]$ defines an endomorphism $\Psi := \sum_{k=0}^{\infty} \Phi_k$.*

Proof. By the definition of summable sequence, each $\Psi^{(j)} = \sum_{k=0}^{\infty} \Phi_k \mod \mathfrak{m}^{j+1}$ is a finite sum of terms, and for $j < i \in \mathbb{N}$

$$\begin{aligned}\pi_{ji}\Psi^{(i)} &= \pi_{ji}\sum_{k=0}^{\infty}\Phi_k \mod \mathfrak{m}^{i+1}\\ &= \sum_{k=0}^{\infty}\Phi_k \mod \mathfrak{m}^{j+1} = \Psi^{(j)}.\end{aligned}$$

Therefore, Ψ is an endomorphism according to the definition of Section 2.3. □

We are particular interested in the case where a sequence of endomorphisms is defined by successive powers $X^k = \underbrace{X \circ \cdots \circ X}_{k}$ of a given derivation X, where $X^k(f)$ means applying the derivative X k-times on $f \in \mathcal{O}_{r,R}[[\mathbf{z}]]$.

Definition 4.1.3. *A derivation $X \in \mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ is called nilpotent if X preserves the ideal $\mathfrak{m}$, i.e. $X(\mathfrak{m}) \subset \mathfrak{m}$, and for all $j \in \mathbb{N}$ exists $N = N(j) \in \mathbb{N}$ such that*

$$X^N(\mathfrak{m}^j) \subset \mathfrak{m}^{j+1},$$

where $\mathfrak{m}^0 = \mathcal{O}_{r,R}[[z]]$. We denote by $\mathcal{N}(\mathcal{O}_{r,R}[[\mathbf{z}]]) \subset \mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ the set of $D_{r,R}$-transversely formal nilpotent derivations.

Proposition 4.1.4. *Let $X \in \mathcal{N}(\mathcal{O}_{r,R}[[\mathbf{z}]])$. For any sequence $\{c_k\}_{k\in\mathbb{N}}$ of complex numbers, the sequence $\{c_k X^k\}_{k\in\mathbb{N}}$ is summable.*

Proof. By the definition, there exist $N(0), N(1), \ldots, N(j) \in \mathbb{N}$, such that

$$X^{N(0)}(\mathfrak{m}^0) \subset \mathfrak{m}, \quad X^{N(1)}(\mathfrak{m}) \subset \mathfrak{m}^2, \ldots X^{N(j)}(\mathfrak{m}^j) \subset \mathfrak{m}^{j+1}.$$

Then,

$$X^{N(0)+\cdots+N(j)}(\mathfrak{m}^0) \subset \mathfrak{m}^{j+1}$$

which implies that the sequence $\{c_k X^k\}_{k\in\mathbb{N}}$ is summable. □

Corollary 4.1.5. *Let $X \in \mathcal{N}(\mathcal{O}_{r,R}[[\mathbf{z}]])$. Then, for all $t \in \mathbb{C}$, the sequence $\left\{\frac{t^k}{k!}X^k\right\}_{k\in\mathbb{N}}$ defines an endomorphism $\exp(tX) = \sum_{k=0}^{\infty}\frac{t^k}{k!}X^k$ called the time t exponential.*

Proof. Taking $c_k = \dfrac{t^k}{k!}$, we conclude from Proposition 4.1.4 that the series $\sum_{k=0}^{\infty} \frac{t^k}{k!} X^k$ converges. $\square$

A consequence of the proof of Proposition 4.1.4 is the following.

Corollary 4.1.6. *For all $i \in \mathbb{N}$, the i^{th}-truncation of the automorphism $\exp(tX)$ is polynomial in $t \in \mathbb{C}$ with coefficients in $\mathcal{O}_{r,R}$. In other words, for $f \in \mathcal{O}_{r,R}[[\mathbf{z}]]$, $\exp(tX)(f) = \left(\sum_{k=0}^{\infty} \frac{t^k}{k!} X^k(f)\right) \mod \mathfrak{m}^{i+1}$ belongs to $\mathcal{O}_{r,R}[t][[\mathbf{z}]]$.*

To study the invertibility of the exponential map, we consider automorphisms and derivations satisfying some flatness conditions. We say that a $D_{r,R}$-transversely formal automorphism $\Phi \in \mathcal{A}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ is *tangent to the identity to order k* if $\Phi(x) = x \mod \mathfrak{m}^{k+1}$ and $\Phi(z_i) = z_i \mod \mathfrak{m}^{k+1}$ for all $i \in \{1, \dots, n\}$. We denote by $\mathcal{A}_k(\mathcal{O}_{r,R}[[\mathbf{z}]])$ the subgroup of such automorphisms.

A derivation $X = a(x, \mathbf{z})\partial_x + \sum_{j=1}^{n} b_j(x, \mathbf{z})\partial_{z_j} \in \mathcal{N}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ is *k-flat* if $a(x, \mathbf{z}) \in \mathfrak{m}^k$, $b_i(x, \mathbf{z}) \in \mathfrak{m}^{k+1}$ for all $i \in \{1, \dots, n\}$. We denote by $\mathcal{N}_k(\mathcal{O}_{r,R}[[\mathbf{z}]])$ for the subalgebra of such derivations.

Proposition 4.1.7 ([Ser09], Theorem 7.2)**.** *For each integer number $k \geq 1$, the exponential map $\exp : \mathcal{N}_k(\mathcal{O}_{r,R}[[\mathbf{z}]]) \to \mathcal{A}_k(\mathcal{O}_{r,R}[[\mathbf{z}]])$; $X \mapsto \exp(X)$ is one-to-one with inverse given by, $\log(\Phi) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} (\Phi - id)^n$.*

We recall that $\mathcal{L}_n$ is the set of n-uples $K \in \{(\mathbb{N}^n - e_1) \cup \dots \cup (\mathbb{N}^n - e_n)\}$, such that $|K| \geq 0$. We denote by $\mathcal{L}_{n,m}$ the set of n-uples $K \in \mathcal{L}_n$, such that $|K| = k_1 + \dots + k_n \geq m$

Example 4.1.8. *Let $\Phi \in \mathcal{A}_k(\mathcal{O}_{r,R}[[\mathbf{z}]])$ correspond to a change of coordinates of the form*

$$(x, z_1, .., z_n) \mapsto (x, \phi_1(x, \boldsymbol{z}), \dots, \phi_n(x, \boldsymbol{z})),$$

where $\phi_i(\boldsymbol{z}) = z_i \mod \mathfrak{m}^{k+1}$. By the Proposition 4.1.7, we have that

$$\log(\Phi) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} (\Phi - id)^n,$$

Since the automorphism Φ maps the variable x to itself, the corresponding derivation $X = \log(\Phi)$ satisfies $X(x) = 0$. Then, we can write X in the following form

$$\begin{aligned} X &= 0\,\partial_{\mathrm{x}} + \sum_{K \in \mathcal{L}_{n,k+1}} b_K(x)\mathbf{z}^K L(\lambda_K) \\ &= \sum_{K \in \mathcal{L}_{n,k+1}} b_K(x)\mathbf{z}^K L(\lambda_K), \end{aligned}$$

where each $b_K \in \mathcal{O}_{r,R}[[\mathbf{z}]]$, and we recall that $L(\lambda) = \sum \lambda_i z_i\, \partial_{\mathrm{z_i}}$, with $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$.

4.2 The Lie Bracket

In this section, we list some results related to the exponential of nilpotent derivations which are going to be used later. After that, we show the connection between the symmetries of a derivation and its centralizer.

In the sequence of propositions, we consider $\Phi \in \mathcal{A}(\mathcal{O}_{r,R}[[\mathbf{z}]])$, $X, Y \in \mathcal{N}(\mathcal{O}_{r,R}[[\mathbf{z}]])$, and $Z, W \in \mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$.

Proposition 4.2.1. *The following properties hold*

1. $\Phi^*[Z,W] = [\Phi^* Z, \Phi^* W]$, *where* $\Phi^* Z = \Phi \circ Z \circ \Phi^{-1}$.
2. $[Z, fW] = f[Z,W] + Z(f)W$, *for* $f \in \mathcal{O}_{r,R}[[\mathbf{z}]]$.
3. *If* $[X,Y] = 0$, *then* $\exp(X+Y) = \exp X \circ \exp Y$.

Proof. The properties (1) and (2) follow directly from the Lie Bracket's definition, and (3) follows from the Newton's Binomial. □

Remark 4.2.2. *Since* $\exp 0 = \mathrm{id}$ *and* $[X, -X] = 0$, *the item* (3) *implies that* $(\exp X)^{-1} = \exp -X$ *and* $\exp(nX) = \exp^n X$, *for all* $n \in \mathbb{Z}$.

The two maps $\mathrm{ad}_Z, \mathrm{Ad}_X : \mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]]) \to \mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ defined by

$$\mathrm{ad}_Z(W) = [Z,W],$$

$$\mathrm{Ad}_X(W) = \exp X \circ W \circ (\exp X)^{-1},$$

are connected by the following well-known result.

Proposition 4.2.3. *For a fixed* $t \in \mathbb{C}$, $\exp(ad_{tX})Z = \mathrm{Ad}_{tX} Z$.

Proof. We give a short proof for the sake of completeness. It is enough to show that

$$\mathrm{ad}_X^n Z = \sum_{k=0}^{n} \binom{n}{k} X^k \circ Z \circ (-X)^{n-k}.$$

We use induction to prove it. The base case $n = 1$ is clear, let us assume that it is true for $n = m$. By hypothesis of induction

$$\begin{aligned}
\mathrm{ad}_X^{m+1} Z &= \mathrm{ad}_X \left(\sum_{k=0}^{m} \binom{m}{k} X^k \circ Z \circ (-X)^{m-k} \right) \\
&= \sum_{k=0}^{m} \binom{m}{k} X^{k+1} \circ Z \circ (-X)^{m-k} - \sum_{k=0}^{m} \binom{m}{k} X^k \circ Z \circ (-X)^{m-k} X \\
&= \sum_{k=0}^{m} \binom{m}{k} X^{k+1} \circ Z \circ (-X)^{m-k} + \sum_{k=0}^{m} \binom{m}{k} X^k \circ Z \circ (-X)^{m-k+1},
\end{aligned}$$

By the Pascal's triangle

$$\sum_{k=0}^{m}\binom{m}{k}X^{k+1}\circ Z\circ(-X)^{m-k}+\sum_{k=0}^{m}\binom{m}{k}X^{k}\circ Z\circ(-X)^{m-k+1}=$$
$$=\sum_{k=0}^{m+1}\binom{m+1}{k}X^{k}\circ Z\circ(-X)^{m-k}.$$

□

Definition 4.2.4 ($D_{r,R}$*-transversely formal symmetry*)**.** *A $D_{r,R}$-transversely formal symmetry for $Z \in \mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ is an automorphism $\Phi \in \mathcal{A}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ such that*

$$\Phi \circ Z \circ \Phi^{-1} = Z.$$

We remark that we can similarly define a formal symmetry for a derivation in $\mathcal{D}(\mathbb{C}[[x,\mathbf{z}]])$. In addition, Proposition 4.1.7 has its version for automorphisms and derivations over $\mathbb{C}[[x,\mathbf{z}]]$. Then, the following results of this section have their respective statements over the formal objects.

Definition 4.2.5 (x*-normalized automorphism*)**.** *A $D_{r,R}$-transversely formal automorphism Φ is said to be x-normalized if it is $\mathcal{O}_{r,R}$-linear and has the form*

$$\Phi(x,\mathbf{z})=\left(x,\sum_{i=1}^{n}a_{1i}z_i+\phi_1(x,\mathbf{z}),\ldots,\sum_{i=1}^{n}a_{ni}z_i+\phi_n(x,\mathbf{z})\right),$$

where $(a_{ij})_{n\times n}$ is an invertible constant matrix and $\phi_1,\ldots,\phi_n \in \mathfrak{m}^2 \subset \mathcal{O}_{r,R}[[\mathbf{z}]]$. We denote by $\mathcal{A}_{norm}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ the set of such automorphisms.

Supposing that the maps $\phi_1,\ldots,\phi_n$ in Definition 4.2.5 lie in the respective rings $\mathcal{O}_R[[\mathbf{z}]]$ and $\mathbb{C}[[x,\mathbf{z}]]$, we define similarly $\mathcal{A}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$ and $\mathcal{A}_{norm}(\mathbb{C}[[x,\mathbf{z}]])$.

In the next result, we use the fact that any x-normalized automorphism $\Phi \in \mathcal{A}_{norm}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ can be uniquely decomposed as $\Phi = A \circ \Psi$, where A is an invertible linear change of coordinates in the indeterminates $x, z_1, \ldots, z_n$, i.e.,

$$Ax = x,$$
$$Az_i=\sum_{j=1}^{n}a_{ij}z_j,\quad a_{ij}\in\mathbb{C}$$

and Ψ is a $D_{r,R}$-transversely formal automorphism tangent to the identity to order 1. Using Proposition 4.1.7, we can write

$$\Phi = A \circ \exp Z,$$

for a uniquely determined $D_{r,R}$-transversely formal nilpotent derivation Z. We say that this decomposition is the *exponential decomposition* of Φ.

Lemma 4.2.6. *Let $X \in \mathcal{D}(\mathbb{C}[[x, \mathbf{z}]])$ be a linear derivation and Φ an x-normalized $D_{r,R}$-transversely automorphism. If Φ is a $D_{r,R}$-transversely formal symmetry for X, and $\Phi = A \circ \exp Z$ is the exponential decomposition of Φ, then the automorphisms A and $\exp Z$ are $D_{r,R}$-transversely formal symmetries for X.*

Proof. Note that the linear part of $\Phi^* X$ is $A^* X$. As X is linear and Φ is a symmetry for X, we have $A^* X = X$. On the other hand, $X = \Phi^* X = \exp Z^*(A^* X) = \exp Z^* X$, then $\exp Z$ is also a symmetry for X. □

For the following sequence of results, consider $X \in \mathcal{N}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ and $Z \in \mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$.

Lemma 4.2.7. *Let $\exp X$ be a $D_{r,R}$-transversely formal symmetry for Z. Then, $\exp tX$ is a $D_{r,R}$-transversely formal symmetry for Z for all $t \in \mathbb{C}$.*

Proof. The following argument is based on the proof of Lemma 2.4.9 [Dia19]. By Corollary 4.1.6, we know that for all $i \in \mathbb{N}$ the i^{th}-truncation of $\exp tX$ is polynomial on t, so the map

$$P_k(t, x, \mathbf{z}) = (\exp tX \circ Z \circ \exp -tX - Z) \mod \mathfrak{m}^{k+1}$$

is as well.

By the Remark 4.2.2, $\exp nX = (\exp X)^n$ for all $n \in \mathbb{N}$. All these facts together imply that

$$\begin{aligned}
\exp nX \circ Z \circ \exp -nX &= \exp^n X \circ Z \circ (\exp X)^{-n} \\
&= \exp^{n-1} X \circ (\exp X \circ Z \circ (\exp X)^{-1}) \circ (\exp X)^{-n+1} \\
&= \exp^{n-1} X \circ Z \circ (\exp X)^{-n+1} \\
&\;\;\vdots \\
&= Z.
\end{aligned}$$

In other words, for all $n, k \in \mathbb{N}$, the polynomial $P_k(n, x, \mathbf{z})$ vanishes. As consequence, $P(t, x, \mathbf{z}) = 0$, therefore, $\exp tX \circ Z \circ \exp -tX = Z$ for all $t \in \mathbb{C}$. The reciprocal is obvious. □

Lemma 4.2.8. *For all $t \in \mathbb{C}$, the exponential map $\exp tX$ is a $D_{r,R}$-transversely formal symmetry for Z if only if $[X, Z] = 0$.*

Proof. Assume that the $\exp tX$ is a $D_{r,R}$-transversely formal symmetry. The Proposition 4.2.3 implies

$$Z = \mathrm{Ad}_{tX} Z = \exp(\mathrm{ad}_{tX})Z = (Z + t[X, Z] + \frac{t^2}{2}[X, [X, Z]] + \dots).$$

Taking the derivative on t and evaluating on zero, we find $[X, Z] = 0$. The reciprocal is obvious. □

Proposition 4.2.9. *The map $\exp X$ is a $D_{r,R}$-transversely formal symmetry for Z if only if X commutes with Z, i.e. $[X, Z] = 0$.*

Proof. By Lemma 4.2.7, $\exp X$ is a $D_{r,R}$-transversely formal symmetry for Z if only if for all $t \in \mathbb{C}$, $\exp tX$ is also a $D_{r,R}$-transversely formal symmetry for Z. By Lemma 4.2.8, the $\exp tX$ is a $D_{r,R}$-transversely formal symmetry for Z for all $t \in \mathbb{C}$ if only if $[X, Z] = 0$. □

4.3 Normal form of x-normalized $\mathbf{D_R}$-transversely Formal Vector Fields

In this section, we adapt the Normal Formal Theory of Poincaré described in Section 3.2 to our present setting of D_R-transversely formal derivations.

Definition 4.3.1 (*x-normalized derivation in $\mathcal{D}(\mathcal{O}[[\boldsymbol{z}]])$*)**.** *An element $X \in \mathcal{D}(\mathcal{O}_R[[\mathbf{z}]])$ is called x-normalized in $\mathcal{D}(\mathcal{O}[[\boldsymbol{z}]])$ if it has the following form*

$$x\partial_x + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} z_j \partial_{z_i} + \sum_{i=1}^{n} b_i(x, \mathbf{z})\partial_{z_i}, \tag{4.2}$$

where $a_{ij} \in \mathbb{C}$ and $b_i(x, \mathbf{z}) \in \mathfrak{m}^2 \subset \mathcal{O}_R[[\mathbf{z}]]$. We denote by $\mathcal{D}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$ the set of x-normalized derivations in $\mathcal{D}(\mathcal{O}_R[[\mathbf{z}]])$.

In the next proof, we use the *graded lexicographic order* $>_{gr.lex}$ on $\mathbb{Z}^n$ defined as follows. Let $K = (k_1, \dots, k_n)$ and $L = (l_1, \dots, l_n)$ be two n-uples in $\mathbb{Z}^n$. We write $K >_{gr.lex} L$ if either $k_1 + \cdots + k_n =: |K| > |L| := l_1 + \cdots + l_n$, or $|K| = |L|$ and $(k_1, \dots, k_n) >_{lex} (l_1, \dots, l_n)$. We say that K is greater than L when $K >_{gr.lex} L$.

We recall that that associated with the diagonal derivation $S = x\,\partial_{\mathrm{x}} + L(\mu)$, where $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$, the respective *negative and positive resonant sets* are given by

$$\begin{aligned} \mathrm{NR}(S) &:= \{T \in \mathcal{L}_n; \langle \mu, T\rangle \in \mathbb{Z}_{\geq 1}\}, \\ \mathrm{PR}(S) &:= \{K \in \mathcal{L}_n \setminus \{\mathbf{0}\}; \langle \mu, K\rangle \in \mathbb{Z}_{\leq 0}\}. \end{aligned}$$

Proposition 4.3.2. *For any derivation $X \in \mathcal{D}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$ with $1, \mu_1, \dots, \mu_n$ as eigenvalues there exists an x-normalized D_R-transversely formal change of coordinates which conjugates X to a derivation of the form*

$$x\partial_x + L(\mu) + \sum_{i=2}^{n} \epsilon_i z_i^{-1} z_{i-1} L(e_i) + \sum_{K} x^{-\langle \mu, K\rangle} \mathbf{z}^K L(\lambda_K), \tag{4.3}$$

where $\mu = (\mu_1, \dots, \mu_n)$, $\epsilon_i \in \{0, 1\}$ is nonzero only if $\mu_i = \mu_{i-1}$, and the last sum on the right-hand side is taken over all indices $|K| \geq 1$ such that $K \in \mathrm{PR}(x\partial_x + L(\mu))$.

Proof. Applying the usual Jordan Normal Theory, we can assume that the linear part of X has the form

$$X_{lin} = x\partial_x + L(\mu) + \sum_{i=2}^{n} \epsilon_i z_i^{-1} z_{i-1} L(e_i),$$

where $\mu = (\mu_1, \ldots, \mu_n)$, $\epsilon_i \in \{0, 1\}$ is nonzero only if $\mu_i = \mu_{i-1}$.

We can write the series expansion of X as

$$X_{lin} + \sum_{K,j} g_{K,j}(x)\mathbf{z}^K L(e_j), \tag{4.4}$$

where the last sum on the right-hand side is taken over all indices $j \in \{1, \ldots, n\}$ and $K \in \mathcal{L}_{n,1}$, and $g_{K,j} \in \mathcal{O}_R$. We say that a nonzero term $g_{K,j}(x)\mathbf{z}^K L(e_j)$ in (4.4) is *resonant* if $\langle \mu, K\rangle$ lies in $\mathbb{Z}_{\leq 0}$ and $g_{K,j}(x) = \lambda_{K,j} x^{-\langle \mu, K\rangle}$, where $\lambda_{K,j} \in \mathbb{C}$.

We eliminate the nonresonant terms by successive applications of automorphisms of the form

$$\Phi = \exp(f(x)\mathbf{z}^K L(e_j)), \tag{4.5}$$

where $f \in \mathcal{O}_R$, $K \in \mathcal{L}_{n,1}$, and $j \in \{1, \ldots, n\}$ are conveniently chosen.

Indeed, we consider the smallest nonresonant term $g_{K_0,j_0}(x)\mathbf{z}^{K_0} L(e_{j_0})$ in (4.4) with respect to the pair (K, j) and to the gr.lex order. The action of $\Phi = \exp\left(f(x)\mathbf{z}^{K_0} L(e_{j_0})\right)$ by conjugation on X gives the expression

$$\begin{aligned}\Phi^* X =& X_{lin} + \sum_{(K,j)\leq(K_0,j_0)} \lambda_{K,j} x^{-\langle \mu, K\rangle} \mathbf{z}^K L(e_j) + \\ &+ (g_{K_0,j_0}(x) - (x\partial_x - \langle \mu, K_0\rangle) f(x))\mathbf{z}^{K_0} L(e_{j_0}) + \mathcal{R},\end{aligned}$$

where the rest term $\mathcal{R}$ is a sum of derivations $g_{K,j}\mathbf{z}^K L(e_j)$ with $(K, j) >_{gr.lex} (K_0, j_0)$.

Writing the series expansion $g_{K_0,j_0}(x) = \sum_{i \neq \langle \mu, K_0\rangle} a_i x^i + a_{\langle \mu, K_0\rangle} x^{\langle \mu, K_0\rangle}$, with the convention that $a_{\langle \mu, K_0\rangle} = 0$ if $\langle \mu, K_0\rangle \in \mathbb{Z}_{\leq 0}$, we can define $f(x) = \sum_{i \neq \langle \mu, K_0\rangle} \frac{a_i}{i + \langle \mu, K_0\rangle} x^i$. As a result, we obtain a new derivation whose smallest nonresonant term is of order strictly greater than (K_0, j_0). □

Notice that the Expression (4.3) is a *normal form* for X according to Definition 3.2.2.

Proposition 4.3.3. *Let $X, Y \in \mathcal{D}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$ and $\Psi \in \mathcal{A}_{norm}(\mathbb{C}[[x, \mathbf{z}]])$. If Ψ conjugates X to Y, then $\Psi \in \mathcal{A}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$.*

Proof. By Proposition 4.3.2, there exist $\Phi_1, \Phi_2 \in \mathcal{A}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$ which diagonalize the respective semisimple parts of X and Y, in other words, we have the respective Jordan decompositions

$$\Phi_1^* X = X_s + X_n = x\partial_x + L(\mu) + X_n,$$

$$\Phi_2^* Y = Y_s + Y_n = x\partial_x + L(\lambda) + Y_n.$$

By hypothesis, there exists $\Psi \in \mathcal{A}_{norm}(\mathbb{C}[[x, \mathbf{z}]])$ such that

$$\Psi^* X = Y,$$

which implies $\mu = \lambda$. By the uniqueness of the Jordan decomposition, the automorphism $\Psi_0 = \Phi_2 \circ \Psi \circ \Phi_1^{-1} \in \mathcal{A}_{norm}(\mathbb{C}[[x, \mathbf{z}]])$ is such that

$$\Psi_0^*(x\partial_x + L(\mu)) = x\partial_x + L(\mu).$$

It means that Ψ_0 is an x-normalized formal symmetry for $x\partial_x + L(\mu)$.

Consider the exponential decomposition $A \circ \exp Z$ of Ψ_0. Since Ψ_0 is x-normalized, by Example 4.1.8, Z has the form $\sum_{K \in \mathcal{L}_{n,1}} b_K(x)\mathbf{z}^K L(\lambda_K)$, where each $b_K \in \mathbb{C}[[x]]$. By Lemma 4.2.6, we know that $\exp Z$ is a symmetry for $x\partial_x + L(\mu)$. And by Proposition 4.2.9, we have that

$$[Z, x\partial_x + L(\mu)] = 0.$$

Using the above expansion of Z, this equality is equivalent to state that

$$(x\partial_x + \langle \mu, K \rangle) b_K(x) = 0$$

for all $K \in \mathcal{L}_{n,1}$. Writing $b_K(x) = \sum_{i=0}^{\infty} c_i x^i$, with $c_i \in \mathbb{C}$, we obtain for each $i \in \mathbb{N}$

$$(i + \langle \mu, K \rangle) c_i = 0.$$

This implies that either $\langle \mu, K \rangle \notin \mathbb{Z}_{<0}$ and $b_K = 0$ or else $b_K(x) = cx^{-\langle \mu, K \rangle}$ for some constant $c \in \mathbb{C}$. In other words, Z is a D_R-transversely formal vector field, and consequently, $\exp Z$ is a D_R-transversely formal change of coordinates. In conclusion, $\Phi = \Phi_2^{-1} \circ \Psi_0 \circ \Phi_1$ is a D_R-transversely formal automorphism. □

The property of no transverse negative resonance stated in the introduction can be reformulated as follows.

Definition 4.3.4. *We say that a x-normalized derivation $X \in \mathcal{D}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$ has no transverse negative resonance if*

$$\langle \mu, \mathcal{L}_n \rangle \cap \mathbb{Z}_{\geq 1} = \emptyset,$$

where $1, \mu_1, \dots, \mu_n$ are the eigenvalues of X and $\mu = (\mu_1, \dots, \mu_n)$.

The definition of an x-normalized derivation can be extended to derivations $X \in \mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$. Indeed, an x-normalized $D_{r,R}$-transversely formal derivation is a derivation which has the form (4.2), where each $b_i(x, \mathbf{z}) \in \mathfrak{m}^2 \subset \mathcal{O}_{r,R}[[\mathbf{z}]]$. We denote by $\mathcal{D}_{norm}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ the set of *x-normalized derivations* in $\mathcal{D}(\mathcal{O}_R[[\mathbf{z}]])$.

The next result is a central step in the proof of the Main Theorem. More specifically, it characterizes x-normalized $D_{r,R}$-transversely formal derivations that commute with an x-normalized D_R-transversely formal derivation with no transverse negative resonance.

Theorem 4.3.5. *Let X be an x-normalized D_R-transversely formal derivation that has no transverse negative resonance and Y be an x-normalized $D_{r,R}$-transversely formal derivation. If $[X, Y] = 0$, then Y is an x-normalized D_R-transversely formal derivation.*

Proof. By Proposition 4.3.2, we can assume that X has the form

$$X = x\partial_x + L(\mu) + \sum_{i=2}^{n} \epsilon_i z_i^{-1} z_{i+1} L(e_i) + \sum_K b_K x^{-\langle \mu, K\rangle} \mathbf{z}^K L(\lambda_K).$$

Since Y is x-normalized, it can be expanded as

$$Y = x\partial_x + \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} z_j \partial_{z_i} + \sum_{l,K} d_{lK} x^l \mathbf{z}^K L(\lambda_K),$$

where $l \in \mathbb{Z}$, $K \in \mathcal{L}_{n,1}$.

As Y commutes with X, it has to commute with its semisimple part $x\partial_x + L(\mu)$. This implies that for each $l \in \mathbb{Z}$, $K \in \mathcal{L}_{n,1}$, we must have

$$(l + \langle \mu, K\rangle) d_{lK} x^l \mathbf{z}^K L(\lambda_K) = [x\partial_x + L(\mu), d_{lK} x^l \mathbf{z}^K L(\lambda_K)] = 0.$$

Since X has no transverse negative resonance, both expressions can vanish only if $l \geq 0$. This means all monomials in the expansion of Y have positive exponents on x. Consequently $Y \in \mathcal{D}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$. □

Chapter 5

Characterization of singular foliations by the holonomy

As showed in Section 1.2, if the holonomies of two regular foliations are analytically conjugated, then it is possible to construct a change of coordinates which sends the leaves of a foliation into the leaves of another using the path lifting method. As a consequence of this result, we will show in Section 5.2 that it is always possible to construct an *x-normalized automorphism* that conjugates local generators of crossing type foliations on a neighborhood of $D_{r,R} \times \{\mathbf{0}\}$ if the conditions *(a)* and *(c)* (see Section Introduction) are satisfied.

At the end of Section 5.2, we show a less restrictive hypothesis over the eigenvalues of the local generators than condition *(a)*. The decision to make *(a)* as main hypothesis was not just to keep the simplicity of the statement of the Main Theorem, but also since this condition does not represent any new restriction to equivalent foliations. Section 5.3 contains the proof of the Main Theorem and its application in dimension 3.

In Section 5.5, we study the necessity of conditions *(a)*, *(b)*, and *(c)*. In particular, we show that condition *(b)* is almost optimal. More precisely, if an x-normalized local generator with semisimple part $S = x\,\partial_{\mathrm{x}} + L(\mu)$, has non empty set $\mathrm{NR}(S)$, then, under some other hypothesis, there are two vector fields satisfying *(a)* and *(c)* having S as semisimple part, but which do not generate analytically equivalent foliations. Moreover, as a consequence of this result, we proved that the conjugation of the holonomies *does not imply* the analytic equivalence of the foliations as in the regular case.

5.1 Fuchsian singularity and the preparation of the linear part

In this subsection, we consider D_R-transversely convergent derivations X which are linear in the $\mathbf{z}$-variables, i.e.,

$$X_{lin} = x\partial_x + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} z_j \partial_{z_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}(x) z_i \partial_{z_j},$$

where $a_{ij} \in \mathbb{C}$, and $b_{ij} \in \mathcal{O}_R$ vanishes at $0 \in \mathbb{C}$. The goal is to present some conditions under which we can eliminate all $b_{ij}(x)$ coefficients by a change of coordinates.

As we will show, this is a classical problem in the theory of linear differential systems of Fuchsian type and the hypothesis that X_{lin} has no transverse negative resonance is precisely the necessary condition. For a detailed discussion on this topic, we refer to [IY08].

Definition 5.1.1 (*Fuchsian singularity*)**.** *A Fuchsian singularity is a system of differential equations of the form*

$$\begin{aligned} \boldsymbol{z}' &= A_0 \boldsymbol{z} + x A_1 \boldsymbol{z} + \dots, \\ \frac{\partial x}{\partial t} &= x, \end{aligned} \tag{5.1}$$

where $A_0, A_1, \dots$ are constant matrices. The matrix A_0 is called the residue of the Fuchsian singularity.

Consider two cylinders $\mathcal{R}\mathrm{S}_1 \times \mathbb{C}^n$ and $\mathcal{R}\mathrm{S}_2 \times \mathbb{C}^n$ over two Riemann surfaces $\mathcal{R}\mathrm{S}_1$ and $\mathcal{R}\mathrm{S}_2$. Each cylinder is naturally equipped with the projection $\pi_1 : \mathcal{R}\mathrm{S}_1 \times \mathbb{C}^n \to \mathcal{R}\mathrm{S}_1$ (resp., $\pi_2 : \mathcal{R}\mathrm{S}_2 \times \mathbb{C}^n \to \mathcal{R}\mathrm{S}_2$) into the base. A *gauge transform* between these two cylinders is a holomorphic map $H : \mathcal{R}\mathrm{S}_1 \times \mathbb{C}^n \to \mathcal{R}\mathrm{S}_2 \times \mathbb{C}^n$ which respects these projections, and it is linear on each fiber $\pi_1^{-1}(a) = \{a\} \times \mathbb{C}^n$, for any $a \in T$. This means that there exists a holomorphic map $h : \mathcal{R}\mathrm{S}_1 \to \mathcal{R}\mathrm{S}_2$ such that

$$\pi_2 \circ H = h \circ \pi_1, \quad H|_{\pi_1^{-1}(a)} \colon \pi_1^{-1}(a) \to \pi_2^{-1}(h(a))$$

Definition 5.1.2 (*Gauge equivalence*)**.** *Two Fuchsian singularities are said to be gauge equivalent if they can be transformed into each other by an invertible gauge transform.*

Definition 5.1.3 (*Resonant Fuchsian singularity*)**.** *A Fuchsian singularity with residue matrix A_0 is resonant, if there are two eigenvalues of A_0 that differ by a positive integer number. Otherwise, the Fuchsian singularity is nonresonant.*

Theorem 5.1.4. *A Fuchsian system with a nonresonant residue matrix A_0 is formally equivalent to the system $t\boldsymbol{z}' = A_0 \boldsymbol{z}$ by a formal gauge transformation.*

Proof. It is a direct consequence of the Poincaré-Dulac Theorem. □

Remark 5.1.5. *All resonant monomials that are linear in $z_1, \dots, z_n$ have the form $x^k z_j \, \partial_{z_i}$. Thus, the only resonances between the eigenvalues $1, \mu_1, \dots, \mu_n$ that can prevent these monomials to be eliminated from (5.1) should have the* *form $\mu_i = \mu_j + k$ with $k \in \mathbb{Z}_{>0}$; all other eventual resonances correspond to monomials that do not appear in (5.1) from the outset.*

In particular, the formal change of coordinates that linearizes the Fuchsian singularity in theorem above is given by successive applications of automorphisms of the form (4.5). In other words, the change of coordinates is x-normalized.

Theorem 5.1.6 ([IY08], Theorem 16.16)**.** *Any formal gauge transformation conjugating two Fuchsian singularities, always converges.*

Now, let $X \in \mathcal{D}(\mathcal{O}_R\{\mathbf{z}\})$ be a derivation having the form

$$X = x\partial_x + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} z_j \partial_{z_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}(x) z_i \partial_{z_j},$$

where $a_{ij} \in \mathbb{C}$, and $b_{ij} \in \mathcal{O}_R$ vanishes at $0 \in \mathbb{C}$. Note that the derivation X is associated with the following Fuchsian system

$$\begin{cases} \frac{\partial x}{\partial t} &= x, \\ \frac{\partial \mathbf{z}}{\partial t} &= (A_0 + xA_1 + \dots)\mathbf{z}, \end{cases} \tag{5.2}$$

where $A_0 = (a_{ij})_{n\times n}$, and A_i is a n-dimensional square complex matrix for all $i \in \mathbb{Z}_{>0}$. Then, from the Fuchsian theory above, we can enunciate the following.

Lemma 5.1.7. *Let $X \in \mathcal{D}(\mathcal{O}_R\{\mathbf{z}\})$ be a derivation having the associated differential equation of the form (5.2). If X has no transverse negative resonance, then, possibly reducing the radius R, there exists an x-normalized change of coordinates which conjugates X to a derivation with the form*

$$x\partial_x + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} z_j \partial_{z_i}.$$

Proof. Let $1, \mu_1, \dots, \mu_n$ be the eigenvalues of X. The hypothesis of no transverse negative resonance implies that $\mu_i - \mu_j \notin \mathbb{Z}_{\geq 1}$, for any $i \neq j$. Hence, the expansion (5.2) has nonresonant residue matrix A_0. Then, by Theorems 5.1.4 and 5.1.6, there exists an x-normalized change of coordinates which conjugates X to a derivation with the same form, but with all coefficients $b_{ij}(x)$ equal to zero. □

5.2 Holonomy and equivalence of x-normalized foliations in a neighborhood of $\mathbb{S}^1$

Lemma 5.2.1. *Let $(\mathcal{F}, H, \Gamma)$ be a crossing type foliation (see Introduction), whose local generators have no transverse negative resonance. Then there exist local coordinates*

$$(x, \mathbf{z}) = (x, z_1, \dots, z_n)$$

such that $\Gamma = \{\mathbf{z} = 0\}$, $H := \{x = 0\}$. *Moreover, once these coordinates are fixed, there exists a unique local generator* X *for* $\mathcal{F}$ *that is an* x*-normalized* D_R*-transversely convergent derivation.*

Proof. We can choose coordinates $(x, \mathbf{z})$ such that $H := \{x = 0\}$, $\Gamma = \{\mathbf{z} = 0\}$. The condition (ii.b) in the definition of crossing type foliation (see Section Introduction) implies that a local generator for $\mathcal{F}$ can be written in these coordinates as

$$X_0 = g(x, \mathbf{z})x\partial_x + \sum_{i=1}^{n} h_i(x, \mathbf{z})\partial_{z_i},$$

where $g, h_1, \ldots,$ and h_n do not have common factors and $h_1, \ldots, h_n$ lie in the ideal $\mathfrak{m}$. By Condition (iii.), we have that $g(0) \neq 0$. Therefore, $X_1 = g^{-1}X_0$ is a local generator of $\mathcal{F}$.

By hypothesis, X_1 has no transverse negative resonance, then, by Lemma 5.1.7, there exists an x-normalized D_R-transversely convergent change of coordinates that conjugates X_1 to

$$X = x\partial_x + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} z_j \partial_{z_i} + \sum_{i=1}^{n} b_i(x, \mathbf{z})\partial_{z_i},$$

where $(a_{ij})_{n\times n}$ is a constant matrix, and $b_i \in \mathfrak{m}^2 \subset \mathcal{O}_R\{\mathbf{z}\}$ for $1 \leq i \leq n$. □

We say that the coordinates given by this lemma are *adapted* to the crossing type foliation $(\mathcal{F}, H, \Gamma)$ and that X is the *x-normalized local generator* for $\mathcal{F}$ in these adapted coordinates.

For any $R > 0$, up to a constant rescaling in the x variable, we can suppose that the x-normalized local generators are defined in a domain containing a neighborhood of $\overline{D_R \times \{\mathbf{0}\}} \subset \mathbb{C}^{n+1}$. Then, we fix once and for all an constant $R > 1$.

In these adapted coordinates, the local Γ-holonomy can be computed by lifting the circular path $\{(e^{2\pi\theta}, 0); \theta \in [0, 1]\}$ along the leaves of the foliation that goes through a small poli-disk $\mathbb{D} \subset \mathbb{C}^n$ transverse to Γ^* at the point $(1, \mathbf{0})$.

Definition 5.2.2 ($D_{r,R}$*-transversely equivalent crossing type foliations*)**.** *We say that* $(\mathcal{F}, H, \Gamma)$ *and* $(\mathcal{G}, L, \Omega)$ *are* $D_{r,R}$*-transversely equivalent if there exist* $0 < r < 1 < R$*, respective adapted coordinates* $(x, \mathbf{z})$ *and* $(y, \mathbf{w})$*, and a bianalytic map* Ψ *between two open neighborhoods* $U, V \subset \mathbb{C}^{n+1}$ *of* $D_{r,R} \times \{\mathbf{0}\}$ *such that* Ψ *conjugates the* x*-normalized and* y*-normalized local generators restricted to* U *and* V*, and we can write* Ψ *in the form*

$$(y, \mathbf{w}) = \left(x, \sum_{i=1}^{n} a_{1i} z_i + \psi_1(x, \mathbf{z}), \ldots, \sum_{i=1}^{n} a_{ni} z_i + \psi_n(x, \mathbf{z})\right),$$

where $(a_{ij})_{n\times n}$ *is an invertible constant matrix and* $\psi_1, \ldots, \psi_n \in \mathfrak{m}^2 \subset \mathcal{O}_{r,R}[[\mathbf{z}]]$.

We observe that Ψ is an x-normalized $D_{r,R}$-transversely automorphism (see Definition 4.2.5). In addition, we can prove that the $D_{r,R}$-transversely equivalence is enough to guarantee conjugated linear parts for the respective local generators.

Claim 5.2.3. *Let $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ be two crossing type foliations whose respective local generators X and Y. If $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ are $D_{r,R}$-transversely equivalent, then X and Y have conjugated linear parts.*

Proof. By the definition of $D_{r,R}$-transversely equivalence, there is a automorphism Ψ, which conjugates the local generators, and, in conveniently chosen adapted coordinates, it has the form

$$(y, \mathbf{w}) = \left(x, \sum_{i=1}^{n} a_{1i} z_i + \psi_1(x, \mathbf{z}), \ldots, \sum_{i=1}^{n} a_{ni} z_i + \psi_n(x, \mathbf{z}) \right),$$

where $A = (a_{ij})_{n \times n}$ is an invertible constant matrix and $\psi_1, \ldots, \psi_n \in \mathfrak{m}^2 \subset \mathcal{O}_{r,R}[[\mathbf{z}]]$. Now, consider the exponential decomposition $B \circ \exp Z$ of Ψ, where B is the block matrix $B = 1 \oplus A$. Since Z is tangent to the identity, the linear part of $(\operatorname{Ad}_Z X)$ is $X \mod \mathfrak{m}^2$. Then, the linear part of $(\Psi^* X)$ is $(A^*(X \mod \mathfrak{m}^2))$. □

Even though the next result is a simple application of Theorem 2, chapter IV [CN85], we are going to give a proof of the new fact that the map which conjugates the local generators is x-normalized. We refer to [CN85] for further details.

Proposition 5.2.4. *Let $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ be two crossing type foliations with respective local generators X and Y verifying (a), (b) and (c) (see Section Introduction). Then, the crossing type foliations are $D_{r,R}$-transversely equivalent.*

Proof. Let $K \subset \Gamma^*$ be a compact set containing the unity circle $\mathbb{S}^1$. Given a point $(x, \mathbf{z})$ in a convenient neighborhood of K, let $\alpha_x : [0, -ln\|x\|] \to K$ and $\beta_x : [0, \tilde{t}] \to K$ be the curves such that $\alpha_x(t) = (xe^t, \mathbf{0})$, $\beta_x(t) = \big(x/\|x\|e^{-2\pi i t}, \mathbf{0}\big)$, and $\beta_x(\tilde{t}) = (1, \mathbf{0})$.

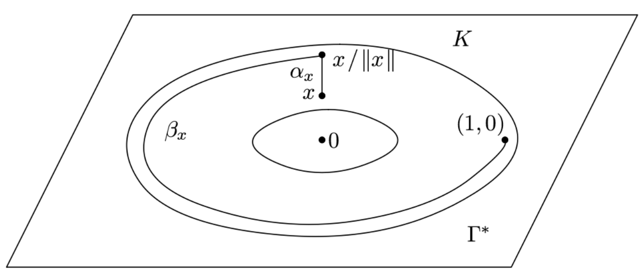


Figure 5.1: Composed path

Denote by $\mathrm{L}^{\gamma}_{\mathcal{F}}$ and $\mathrm{L}^{\gamma}_{\mathcal{G}}$ the respective lifts of $\gamma \subset K$ to the leaves of $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$, and by φ the map that conjugates the Γ-holonomy to the Ω-holonomy. From Theorem 1.2 [CN77], we recall that the map that conjugates the local generators X and Y is given by

$$\Psi(x, \mathbf{z}) := L^{\alpha_x^{-1}}_{\mathcal{G}} \circ L^{\beta_x^{-1}}_{\mathcal{G}} \circ \varphi \circ L^{\beta_x}_{\mathcal{F}} \circ L^{\alpha_x}_{\mathcal{F}}(x, \mathbf{z}).$$

By the Lemma 5.1.7, we can assume that $X \mod \mathfrak{m}^2 = Y \mod \mathfrak{m}^2 = x\partial_x + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} z_j \partial_{z_i}$, where $A = (a_{ij})_{n\times n}$ is a constant matrix. As consequence, the holonomies have the same linear part. Hence, the map φ has the form $\varphi(x, \mathbf{z}) = (x, \mathbf{z}) \mod \mathfrak{m}^2$.

The restrictions of X to the curves α_x and β_x result into the equivalent differential equations given below, where γ_1 and γ_2 are the respective solutions

$$\begin{cases} \frac{\partial \mathbf{z}}{\partial t} = A\mathbf{z} \mod \mathfrak{m}^2 \\ \mathbf{z}(0) = \mathbf{z} \\ \gamma_1(t) = \exp(tA)\mathbf{z} \mod \mathfrak{m}^2 \end{cases}, \quad \begin{cases} \frac{\partial \mathbf{z}}{\partial t} = -2\pi i A\mathbf{z} \mod \mathfrak{m}^2 \\ \mathbf{z}(0) = \mathbf{z} \\ \gamma_2(t) = \exp(-2\pi i tA)\mathbf{z} \mod \mathfrak{m}^2 \end{cases}.$$

Therefore, there exists a constant $\sigma \in \mathbb{C}$ such that the compositions of the lifts are given by

$$L^{\beta_x}_{\mathcal{F}} \circ L^{\alpha_x}_{\mathcal{F}}(x, \mathbf{z}) = (1, \exp(\sigma A)\mathbf{z}) \mod \mathfrak{m}^2,$$
$$L^{\alpha_x^{-1}}_{\mathcal{G}} \circ L^{\beta_x^{-1}}_{\mathcal{G}}(1, \mathbf{z}) = (x, \exp(-\sigma A)\mathbf{z}) \mod \mathfrak{m}^2.$$

We conclude that $\Psi(x, \mathbf{z}) = (x, \mathbf{z}) \mod \mathfrak{m}^2$ and $\Psi(x, 0) = (x, 0)$, in other words, the map Ψ is an x-normalized automorphism. □

Remark 5.2.5. *Looking at the structure of the logarithm of invertible linear maps, one sees that the condition (a) of the Main Theorem can be replaced by the following nonequivalent condition: Writing the respective semisimple parts of X and Y as $x\partial_x + L(\mu)$ and $x\partial_x + L(\lambda)$, no difference $\mu_i - \lambda_j$, $1 \le i, j \le n$ is a nonzero integer. Indeed, first let us prove the following.*

Claim 5.2.6. *Let $A, B \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ be two linear endomorphisms such that no eigenvalue of A differs from eigenvalues of B by a nonzero multiple of $2i\pi$. Suppose further that $\exp A$ and $\exp B$ are conjugated. Then A and B are conjugated.*

Proof. By hypothesis, $\exp A$ and $\exp B$ have the same set of eigenvalues. In addition, there exists an invertible endomorphism C mapping each generalized eigenspaces V_λ of $\exp A$ to the generalized eigenspaces U_λ of $\exp B$. Restricted to each eigenspace V_λ (resp. U_λ), we can write

$$A_\lambda = P_\lambda((\exp A)_\lambda), \quad B_\lambda = Q_\lambda((\exp B)_\lambda),$$

where P_λ, Q_λ are the Taylor expansion of the chosen branch of the logarithm map centered on λ.

By the hypothesis on the eigenvalues of A and B, the same branch of logarithm is chosen in both Taylor expansions. That is, $P_\lambda = Q_\lambda$. Therefore, we obtain

$$A_\lambda = P_\lambda((\exp A)_\lambda) = P_\lambda(C(\exp B)_\lambda C^{-1}) = CP_\lambda((expB)_\lambda)C^{-1} = CB_\lambda C^{-1}.$$

□

As a consequence, we can enunciate the following.

Corollary 5.2.7. *Let $A, B \in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^n)$ be two linear endomorphisms such that $\exp 2i\pi A$ and $\exp 2i\pi B$ are conjugated, and no eigenvalue of A differs from an eigenvalue of B by a nonzero integer. Then A and B are conjugated.*

The following example shows that even if A and B have the same set of eigenvalues and the $\exp A$ and $\exp B$ are conjugated, we can not guarantee that A and B are conjugated.

Example 5.2.8. *Consider the matrices*

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2i\pi & 1 \\ 0 & 0 & 0 & 2i\pi \end{pmatrix}, \quad B = \begin{pmatrix} 2i\pi & 0 & 0 & 0 \\ 0 & 2i\pi & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then we have that $\exp A = \exp B$. Furthermore, A and B have the same set of eigenvalues $\{0, 2i\pi\}$ with the same algebraic multiplicity but A and B are not conjugated.

5.3 Proof of Main Theorem and its corollary in dimension 3

Proof of the Main Theorem. Let $\Psi \in \mathcal{A}_{norm}(\mathcal{O}_{r,R}\{\mathbf{z}\})$ be the automorphism defined by Proposition 5.2.4 which conjugates X to Y, the respective local generators, in a neighborhood of an annulus $D_{r,R} \times \{\mathbf{0}\}$.

By Proposition 4.3.2, there exist $\Phi_1, \Phi_2 \in \mathcal{A}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$ which diagonalize the respective semisimple parts of X and Y, in other words

$$\Phi_1^* X = X_s + X_n,$$

$$\Phi_2^* Y = Y_s + Y_n,$$

where $X_s = Y_s = x\partial_x + L(\mu)$. By the uniqueness of the Jordan decomposition, the automorphism $\Psi_0 = \Phi_2 \circ \Psi \circ \Phi_1^{-1} \in \mathcal{A}_{norm}(\mathcal{O}_{r,R}[[\mathbf{z}]])$ is such that

$$\Psi_0^*(x\partial_x + L(\mu)) = x\partial_x + L(\mu).$$

It means that Ψ_0 is an x-normalized $D_{r,R}$-transversely formal symmetry for $x\partial_x + L(\mu)$.

Let $A \circ \exp Z$ be the exponential decomposition of Ψ_0 (see section 4.2). Applying Lemma 4.2.6, the $\exp Z$ is a symmetry for $x\partial_x + L(\mu)$, and by Proposition 4.2.9, we must have

$$[Z, x\partial_x + L(\mu)] = 0.$$

Since $x\partial_x + L(\mu)$ has no transverse negative resonance, we can apply Theorem 4.3.5 to guarantee that $Z \in \mathcal{D}(\mathcal{O}_R[[\mathbf{z}]])$, and then, $\exp Z \in \mathcal{A}(\mathcal{O}_R[[\mathbf{z}]])$.

Finally, we have that $\Psi_0 = A \circ \exp Z \in \mathcal{A}(\mathcal{O}_R[[\mathbf{z}]])$. As consequence, the automorphism Ψ lies in the intersection $\mathcal{A}(\mathcal{O}_R[[\mathbf{z}]]) \cap \mathcal{A}(\mathcal{O}_{r,R}\{\mathbf{z}\})$. Applying Lemma 2.2.3 to the components $\Psi_1, \ldots, \Psi_{n+1}$ of Ψ, we conclude that they lie in $\mathcal{O}_R\{\mathbf{z}\}$. Therefore, $\Psi \in \mathcal{A}_{norm}(\mathcal{O}_R\{\mathbf{z}\})$. □

Remark 5.3.1. *As pointed out in the Introduction, we can reformulate the Main Theorem as follows: If the eigenvalues of $(\mathcal{F}, H, \Gamma)$ satisfy the no transverse negative resonance condition, then $(\mathcal{F}, H, \Gamma)$ is analytically classified by its linear part and its Γ-holonomy.*

In dimension three, we can be more precise about the local classification in terms of the linear part and the holonomy proving Corollary B.

Proof of Corollary B. We recall that the triple $(1, \lambda, \mu)$ is in the Siegel (resp. Poincaré) domain if $0 \in \mathbb{C}$ belong (resp. does not belong) to the the convex hull $\mathcal{K}(1, \lambda, \mu)$.

Up to a symmetry, we can assume that $\mathrm{Im}(\lambda) \geq 0$. Then the triple $(1, \lambda, \mu)$ is in the Siegel domain if and only if the third eigenvalue μ lies in the closed region $\mathfrak{R} := \{z \in \mathbb{C}; \pi \leq \arg z \leq \pi + \arg \lambda\}$ (see figure 5.2).

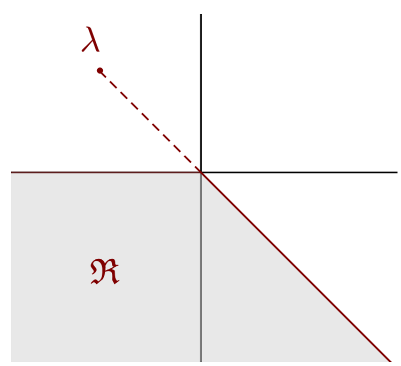


Figure 5.2: Siegel domain

Assume that μ lies in $\mathfrak{R}$. Then two cases can occur:

i. λ lies in $\mathbb{C}\backslash\mathbb{R}$. Then we claim that (λ, μ) satisfies the no negative resonance condition. To show this, we need to prove that the discrete positive cone

$$\mathcal{C} = \{p_1\lambda + p_2\mu; p_1, p_2 \in \mathbb{Z}_{\geq -1}, p_1 + p_2 \geq 0\}$$

contains no element $n \in \mathbb{Z}_{\geq 1}$. We consider initially the case where $\mu = 0$. Then $\mathcal{C} = \{p_1\lambda; p_1 \in \mathbb{Z}_{\geq -1}\}$ and the equality $\mathrm{Im}(p_1\lambda) = \mathrm{Im}(n)$ implies that $p_1 = 0$. Consequently, we have that $0 = \mathrm{Re}(p_1\lambda) = n$. Hence $\mathcal{C} \cap \mathbb{Z}_{\geq 1} = \varnothing$.

Now, assume that μ is nonzero. Since, $\mu \in \mathfrak{R}$, the imaginary part of μ is negative. Remark that the euclidean inner product of the vectors in $\mathbb{R}^2$ associated to two complex numbers z, w is given by $\mathrm{Re}(z\overline{w})$, and iz and $-iz$ are orthogonal to z. Hence, the set $\mathcal{C}$ lies in the half-plane

$$H = \{z; \mathrm{Re}(z\overline{i(\lambda - \mu)}) \geq 0\}$$

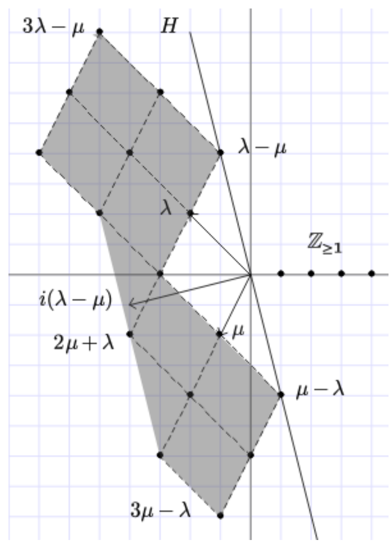


Figure 5.3: The discrete positive cone $\mathcal{C}$

Now, for an arbitrary positive integer number n,

$$\mathrm{Re}(n\overline{i(\lambda - \mu)}) = n\,\mathrm{Re}(\overline{i(\lambda - \mu)}) = -n\,\mathrm{Im}(\lambda - \mu) < 0$$

which shows that $n \notin H$.

ii. λ lies in $\mathbb{R}$. Then the region $\mathfrak{R}$ reduces either to the half nonpositive real line (if $\lambda > 0$) or to the whole complex plane (if $\lambda \leq 0$). We consider these two cases separately. But firstly, remark that the no transverse negative resonance property is equivalent to the conjunction of the following three conditions,

$$\begin{aligned} &\forall p_1, p_2 \in \mathbb{Z}_{\geq 0}, \forall q \in \mathbb{Z}_{\geq 1}; \quad && p_1\lambda + p_2\mu \neq q \qquad (\star) \\ &\forall p, q \in \mathbb{Z}_{\geq 1}; && p\lambda \neq \mu + q \qquad (\star\star) \\ &\forall p, q \in \mathbb{Z}_{\geq 1}; && p\mu \neq \lambda + q \qquad (\star\star\star) \end{aligned}$$

(a) $\mu \leq 0 < \lambda$. Here, $p\mu \leq 0$ and $\lambda + q > 0$, then the condition $(\star\star\star)$ always holds. The negation of the condition $(\star\star)$ corresponds to the equation (2), and the negation of the condition $(\star)$ is equivalent, up to a permutation of coordinates, to $p_1 \geq 1$ and $\lambda = q/p_1 - p_2/p_1\mu \in \mathbb{Q}_{>0} - \mu\mathbb{Q}_{\geq 0}$.

(b) $\lambda \leq 0$. The case where $\mu \neq \mathbb{R}$ is treated in (i.). By a changing coordinates, the case where $\mu > 0$ is the item (ii.a). Hence, up to a permutation of coordinates, just $\mu \leq \lambda \leq 0$ remains to be considered. In this case, the conditions $(\star)$ and $(\star\star\star)$ always hold. The negation of the condition $(\star\star)$ implies that $\mu < \lambda \leq 0$ and that (2) holds.

□

5.4 The $(x, \mathbf{z})$-preserving Endomorphisms

Definition 5.4.1 ($(x,\mathbf{z})$-*preserving formal automorphism*)**.** *An $(x, \boldsymbol{z})$-preserving formal automorphism is a formal automorphism $\Phi \in \mathcal{A}(\mathbb{C}[[x, \boldsymbol{z}]])$ such that*

1. $\Phi(\langle x\rangle) \subset \langle x\rangle$ *and*
2. $\Phi(\langle z_1, \ldots, z_n\rangle) \subset \langle z_1, \ldots, z_n\rangle$.

Geometrically, an $(x, \mathbf{z})$-preserving formal automorphism preserves the formal hypersurface $H := \{x = 0\}$ and the formal curve $\Gamma := \{z = 0\}$. We define similarly an $(x, \mathbf{z})$-*preserving formal derivation* $X \in \mathcal{D}(\mathbb{C}[[x, \mathbf{z}]])$.

Lemma 5.4.2. *Consider a formal automorphism $\Phi = \exp W \in \mathcal{A}(\mathbb{C}[[x, \boldsymbol{z}]])$. Then Φ is an $(x, \boldsymbol{z})$-preserving formal automorphism if and only if W is an $(x, \boldsymbol{z})$-preserving derivation.*

Proof. The proof follows easily from the formal definition of the exponential of a derivation. □

In particular, an $(x, \mathbf{z})$-preserving derivation is given by a formal sum of monomial derivations of the form

$$x^u \mathbf{z}^M x\, \partial_{\mathrm{x}}, \tag{5.3}$$

where $u \in \mathbb{Z}_{\geq 0}$ and $M \in \mathbb{Z}^n_{\geq 0}$ and

$$x^v \mathbf{z}^N L(\lambda), \tag{5.4}$$

with $v \in \mathbb{Z}_{\geq 0}$, $N \in \mathcal{L}_n$.

From now on, all derivations that we shall consider are $(x, \mathbf{z})$-preserving.

Definition 5.4.3 (*Order of a formal series*)**.** *Consider a formal series $f(x, \boldsymbol{z}) = \sum_{(i,K)\in\mathbb{Z}^{n+1}_{\geq 0}} a_{i,K} x^i \boldsymbol{z}^K \in \mathbb{C}[[x, \boldsymbol{z}]]$. We say that the order of f is the smallest number* $\mathrm{ord}(f) := |(i_0, K_0)| \in \mathbb{Z}_{\geq 0}$ *such that $a_{i,K} = 0$, for all $|(i, K)| < |(i_0, K_0)|$.*

Let $S = x\,\partial_{\mathrm{x}} + L(\mu)$ be a diagonal derivation, $Y = S + N$ be a derivation already in a normal form (i.e., N is nilpotent and commutes with S), and u be a unit. In addition, we assume for simplicity that $u(0) = 1$, i.e., we can write $u = 1 + u_1$, where $\mathrm{ord}(u_1) \geq 1$. Hence we can write

$$uY = S + u_1 S + (1 + u_1)N. \tag{5.5}$$

The next lemma shows that there exists a reduction to a normal form keeping essentially the same structure of (5.5).

Lemma 5.4.4. *There exists an $(x, \boldsymbol{z})$-preserving formal automorphism conjugating $u(S + N)$ to the normal form*

$$S + gS + (1 + h)N,$$

where g and h are formal series containing only resonant monomials such that $\mathrm{ord}(g)$ *and* $\mathrm{ord}(h)$ *are greater or equal than 1.*

Proof. Let us write $uY = S + gS + (1 + h)N$ (assuming initially $g = h = u_1$). We consider the conjugation by a monomial automorphism of the form

$$exp(x^a \mathbf{z}^K(\alpha S + \beta N)),$$

where $\alpha, \beta \in \mathbb{C}$ are constants to be chosen. In the sequel notice that

$$[x^a\mathbf{z}^K S, S + gS + (1 + h)N] = -(a + \langle \mu, K\rangle)x^a\mathbf{z}^K S + r_0 S + s_0 N$$
$$[x^a\mathbf{z}^K N, S + gS + (1 + h)N] = -(a + \langle \mu, K\rangle)x^a\mathbf{z}^K N + r_1 S + s_1 N$$

where r_0, r_1, s_0, s_1 denote formal functions of order strictly higher than $x^a\mathbf{z}^K$. Therefore, following the same argument of the Poincaré-Dulac Theorem, by conveniently choosing the constants α, β, we can successively eliminate all monomials in the expansion of g and h which are non-resonant. □

5.5 The almost optimal no transverse positive resonance property

The following result is an adapted version of the Division Lemma of Rham-Saito, see for instance [CCD13], Proposition 1.14.

Lemma 5.5.1. *Consider two derivations $X, Y \in \mathcal{D}(\mathcal{O}_R\{\mathbf{z}\})$ and the respective foliations $\mathcal{F}$ and $\mathcal{G}$ generated by X and Y. Suppose that the singular sets of X and Y have codimension greater or equal than 2, and that there exists an automorphism $\Phi \in \mathcal{A}(\mathcal{O}_R\{\mathbf{z}\})$ taking leaves of $\mathcal{F}$ into leaves of $\mathcal{G}$. Then there exists a unit $u \in \mathcal{O}_R\{\mathbf{z}\}$ such that*

$$\Phi^* Y = uX.$$

Proof. From the hypotheses, the derivations X and Φ^*Y are linearly dependent at each point of their common domain of definition. In other words, we can write

$$a(x,\mathbf{z})\Phi^*Y + b(x,\mathbf{z})X = 0,$$

where $a(x,\mathbf{z})$ and $b(x,\mathbf{z})$ are complex functions.

Denote the derivations by $\Phi^*Y = g_1(x,\mathbf{z})\,\partial_{\mathrm{x}} + \sum_{i=1}^n g_{i+1}(x,\mathbf{z})\,\partial_{\mathrm{z_i}}$ and $X = f_1(x,\mathbf{z})\,\partial_{\mathrm{x}} + \sum_{i=1}^n f_{i+1}(x,\mathbf{z})\,\partial_{\mathrm{z_i}}$. Let Σ_X be the singular set of X and $p \notin \Sigma_X$. Since X does not vanishes at p, there exists an index i such that $f_i(x,\mathbf{z})$ does not vanish in a neighborhood V of p. Then we can define the following analytic function in V

$$u(x,\mathbf{z}) = -\frac{b(x,\mathbf{z})}{a(x,\mathbf{z})} = \frac{g_i(x,\mathbf{z})}{f_i(x,\mathbf{z})}.$$

Since $g_j/f_j = -b/a = g_i/f_i$, for every i,j where f_j, f_i do not vanish, we can define u in $\{D_R \times U\} \setminus \Sigma_X$, where $U \subset \mathbb{C}^n$ is a neighborhood of the origin. By the hypothesis on the codimension of the singular sets, we can extend u to $D_R \times U$. As a result, we can write $Y = uX$. Finally, the hypothesis that set of singular points of Y has codimension greater or equal 2 guarantees that u is a unit. □

Remark 5.5.2. *Given an analytic automorphism $\Phi \in \mathcal{A}(\mathcal{O}_R\{\mathbf{z}\})$, its exponential decomposition*

$$\Phi = A \circ \exp W$$

is such that W is a formal derivation and has a monomial expansion of the form

$$W = \sum_{(a,B)\in\mathcal{L}_{n+1}} x^a \boldsymbol{z}^B(\alpha_{a,B}x\,\partial_{\mathrm{x}} + L(\gamma_{a,B})), \tag{5.6}$$

where $\alpha_{a,B} \in \mathbb{C}$ and $\beta_{a,B} \in \mathbb{C}^n$. We remark that if a derivation W is analytic then the automorphism $\exp W$ is also analytic. On the other hand, the logarithm of an analytic automorphism may not be an analytic derivation.

Lemma 5.5.3. *Let A be an invertible diagonal matrix, and $\Phi \in \mathcal{A}(\mathbb{C}[[x,\boldsymbol{z}]])$ be an $(x,\boldsymbol{z})$-preserving formal automorphism whose exponential decomposition has the form $A \circ \exp W$. Then the formal derivation W is $(x,\boldsymbol{z})$-preserving formal derivation.*

Proof. Since $\exp W = A^{-1} \circ \Phi$ is an $(x,\mathbf{z})$-preserving formal automorphism, the result follows from Lemma 5.4.2. □

Lemma 5.5.4. *Consider $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$, where $\mu_i \neq \mu_j$, for $i \neq j$, and the matrix $A = \mathrm{diag}(\mu_1, \ldots, \mu_n)$. If a matrix B commutes with A, then B is also diagonal.*

Proof. Let $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$. By hypothesis, B commutes with A, i.e., $BA = AB$. Then for all $1 \le i, j \le n$, we have that

$$\sum_{k=1}^{n} b_{ik}a_{kj} = \sum_{k=1}^{n} a_{ik}b_{kj}$$
$$\therefore b_{ij}\mu_j = \mu_i b_{ij}.$$

By hypothesis $\mu_i \neq \mu_j$, then the equation above implies that $b_{ij} = 0$, for all $i \neq j$. □

We recall that the negative and positive resonant sets of a diagonal derivation $S = x\partial_x + L(\mu)$ are given by

$$\mathrm{NR}(S) := \{T \in \mathcal{L}_n; \langle \mu, T\rangle \in \mathbb{Z}_{\ge 1}\},$$
$$\mathrm{PR}(S) := \{K \in \mathcal{L}_n \setminus \{\mathbf{0}\}; \langle \mu, K\rangle \in \mathbb{Z}_{\le 0}\}.$$

Lemma 5.5.5. *Let $S = x\,\partial_{\mathrm{x}} + L(\mu)$, and*

$$N_1 = x^{-\langle \mu, K\rangle}\boldsymbol{z}^K L(\alpha), \quad N_2 = x^{-\langle \mu, K+T\rangle}\boldsymbol{z}^{K+T} L(\beta),$$

where $(T, K) \in \mathrm{NR}(S) \times \mathrm{PR}(S)$ and $\langle \mu, T + K\rangle \le 0$. Then there exists no $(x, \boldsymbol{z})$-preserving formal automorphism conjugating the normal forms

$$X = S + N_1 \text{ and } Y = S + gS + (1 + h)(N_1 + N_2),$$

where g, h are resonant formal series of order greater or equal than 1.

Proof. Suppose by contradiction that there exists an $(x, \mathbf{z})$-preserving formal automorphism Φ conjugating X with Y. By the invariant the Jordan decomposition, we have $\Phi \circ S \circ \Phi^{-1} = S$, and consequently

$$\Phi \circ N_1 \circ \Phi^{-1} = gS + (1 + h)(N_1 + N_2).$$

Since Φ is a symmetry of S, writing $\Phi = A \exp W$, by Lemma 4.2.6, $\exp W$ is symmetry for S. Then we conclude that each monomial derivation in the expansion of W should be either of the form (5.3) with

$$u + \langle \mu, M\rangle = 0,$$

or of the form (5.4) with

$$v + \langle \mu, N\rangle = 0.$$

By computing the respective first Lie brackets, we obtain

$$\begin{aligned}[x^u\mathbf{z}^M x\,\partial_{\mathrm{x}}, N_1] &= [x^u\mathbf{z}^M x\,\partial_{\mathrm{x}}, x^{-\langle \mu, K\rangle}\mathbf{z}^K L(\alpha)] \\ &= -x^{u-\langle \mu, K\rangle}\mathbf{z}^{M+K}(\langle \mu, K\rangle L(\alpha) + \langle \mu, M\rangle x\,\partial_{\mathrm{x}})\end{aligned}$$

and

$$\begin{aligned}[x^v \mathbf{z}^M L(\lambda), N_1] &= [x^v \mathbf{z}^M x L(\lambda), x^{-\langle \mu, K\rangle} \mathbf{z}^K L(\alpha)] \\ &= x^{v-\langle \mu, K\rangle} \mathbf{z}^{N+K} L(\gamma).\end{aligned}$$

In both cases, the exponents of the resulting monomials in the x-variable are greater or equal than $-\langle \mu, K\rangle$. Then by induction, we proof that this property holds for all monomial derivations in the expansion of $\Phi \circ N_1 \circ \Phi^{-1}$.

On the other hand, since $L(\beta)$ and $x\,\partial_{\mathrm{x}} + L(\mu)$ are linearly independent, there exists a monomial of the form

$$x^{-\langle \mu, T+K\rangle} \mathbf{z}^{K+T}(cx\,\partial_{\mathrm{x}} + L(\gamma)), \quad (c, \gamma) \neq (0, 0),$$

in the expansion of Y. By Lemma 5.5.4, the linear automorphism A is a diagonal matrix, then it preserves the exponents of x. Finally, since $-\langle \mu, T+K\rangle < -\langle \mu, K\rangle$, it follows from the previous paragraph that such monomial can not be in the expansion of $\Phi \circ N_1 \circ \Phi^{-1}$. This contradiction shows that Φ can not exist. □

Proposition 5.5.6. *Consider two crossing type foliations* $(\mathcal{F}, H, \Gamma)$ *and* $(\mathcal{G}, L, \Omega)$ *whose respective x-normalized local generators have the form*

$$X = x\,\partial_{\mathrm{x}} + L(\mu) + x^{-\langle \mu, K\rangle} \mathbf{z}^K L(\mu) \quad \textit{and} \tag{5.7}$$

$$Y = x\partial_x + L(\mu) + x^{-\langle \mu, K\rangle} \mathbf{z}^K L(\mu) + x^{-(\langle \mu, K\rangle + \langle \mu, T\rangle)} \mathbf{z}^{K+T} L(\lambda), \tag{5.8}$$

where $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$ *such that* $\mu_i \neq 1$*, for* $i \in \{1, \ldots, n\}$*, and* $\mu_i \neq \mu_j$*, for* $i \neq j$*. If the pair* $(T, K) \in \mathrm{NR}(S) \times \mathrm{PR}(S)$*, where* $S = x\partial_x + L(\mu)$*, satisfies* $\langle \mu, T\rangle + \langle \mu, K\rangle \leq 0$*, then* $(\mathcal{F}, H, \Gamma)$ *and* $(\mathcal{G}, L, \Omega)$ *are not analytically equivalent.*

Proof. Suppose by contradiction that $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ are analytically equivalent. By Lemma 5.5.1, there exists an $(x, \mathbf{z})$-preserving automorphism conjugating X to uY, where u is a unit. Denoting by $N_1 = x^{-\langle \mu, K\rangle} \mathbf{z}^K L(\mu)$ and $N_2 = x^{-(\langle \mu, K\rangle + \langle \mu, T\rangle)} \mathbf{z}^{K+T} L(\lambda)$, and using Lemma 5.4.4, we can assume that there exists a formal $(x, \mathbf{z})$-preserving automorphism conjugating the normal forms

$$X = S + N_1 \text{ and } Y = S + gS + (1+h)(N_1 + N_2).$$

By Lemma 5.5.5, we conclude that no such automorphism can exist. Therefore, $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ are not analytically equivalent. □

Definition 5.5.7 ($D_{r,R}$*-transversely convergent conjugation*)**.** *We say that two derivations* $X, Y \in \mathcal{D}(\mathcal{O}_R\{\mathbf{z}\})$ *are* $D_{r,R}$*-transversely convergent conjugated if there exist two neighborhoods* U *and* V *of* $D_{r,R} \times \{0\}$ *in* $D_{r,R} \times \mathbb{C}^n$ *and a automorphism* $\Phi : U \to V$ *such that*

$$\Phi^* X = Y. \tag{5.9}$$

for all $(x, \boldsymbol{z}) \in U$

Proof of Proposition C. Consider the derivations

$$X = x\,\partial_{\mathrm{x}} + L(\mu) + x^{-\langle\mu,K\rangle}\mathbf{z}^K L(\mu) \quad \text{and}$$
$$V = x^{-\langle\mu,T\rangle}\mathbf{z}^T L(\lambda).$$

Conjugating X by the automorphism $\exp V$, we find

$$\begin{aligned} Y &= \exp(\mathrm{ad}_V)(x\partial_x + L(\mu)) + \exp(\mathrm{ad}_V)(x^{-\langle\mu,K\rangle}\mathbf{z}^K L(\mu)) \\ &= x\partial_x + L(\mu) + x^{-\langle\mu,K\rangle}\mathbf{z}^K L(\mu) - \langle\mu,T\rangle x^{-(\langle\mu,K\rangle+\langle\mu,T\rangle)}\mathbf{z}^{K+T}L(\lambda). \end{aligned}$$

Indeed, we have that $\exp(\mathrm{ad}_V)(x\partial_x + L(\mu)) = x\partial_x + L(\mu)$. In addition, by the hypothesis $\langle\lambda,K\rangle = 0$, we find

$$\begin{aligned} \mathrm{ad}_V(x^{-\langle\mu,K\rangle}\mathbf{z}^K L(\mu)) &= x^{-(\langle\mu,K\rangle+\langle\mu,T\rangle)}\mathbf{z}^{K+T}(\langle\lambda,K\rangle L(\mu) - \langle\mu,T\rangle L(\lambda)) \\ &= -\langle\mu,T\rangle x^{-(\langle\mu,K\rangle+\langle\mu,T\rangle)}\mathbf{z}^{K+T}L(\lambda), \end{aligned}$$

and

$$\begin{aligned} &\mathrm{ad}_V^2(x^{-\langle\mu,K\rangle}\mathbf{z}^K L(\mu)) = \\ &\qquad = -\langle\mu,T\rangle[x^{-\langle\mu,T\rangle}\mathbf{z}^T L(\lambda), x^{-(\langle\mu,K\rangle+\langle\mu,T\rangle)}\mathbf{z}^{K+T}L(\lambda)] \\ &\qquad = -\langle\mu,T\rangle(\langle\lambda,K+T-T\rangle)x^{-\langle\mu,K\rangle-2\langle\mu,T\rangle}\mathbf{z}^{K+2T}L(\lambda) \\ &\qquad = 0. \end{aligned}$$

Notice that, for all $0 < r < R$, $\exp V$ establishes a $D_{r,R}$-conjugation between X and Y (which is not analytic at $x = 0$). In particular, the holonomies of the crossing type foliations $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$, associated respectively to X and Y, are conjugated. On the other hand, it follows from Proposition 5.5.6 that $(\mathcal{F}, H, \Gamma)$ and $(\mathcal{G}, L, \Omega)$ cannot be equivalent.

□

For both following examples, we keep the notation of Proposition C.

Example 5.5.8. *Consider $\mu = (2,-2)$, $\lambda = (1,0)$, $T = (1,0) \in \mathrm{NR}(S)$, and $K = (0,1) \in \mathrm{PR}(S)$. Then we have*

$$\begin{aligned} V &= x^{-2}y\,\partial_{\mathrm{y}}, \\ X &= x\,\partial_{\mathrm{x}} + 2(1+x^2z)(y\,\partial_{\mathrm{y}} - z\,\partial_{\mathrm{z}}), \ \textit{and} \\ Y &= x\,\partial_{\mathrm{x}} + 2(1+x^2z)(y\,\partial_{\mathrm{y}} - z\,\partial_{\mathrm{z}}) - 2y^2z\,\partial_{\mathrm{y}}. \end{aligned}$$

By proposition C, the crossing type foliations generated by X and Y have conjugated holonomies but are not orbitally analytically equivalent.

The following example exhibits a large family of vector fields lying in the class (3.a) of the Corollary B.

Example 5.5.9. *Let $\mu = (-1, -n)$ for some $n \geq 2$ and $m, k \in \mathbb{Z}$ such that $1 \leq m < n$, and $k \geq n - m$. Taking $\lambda = (1, 0)$, and $T = (m, -1), K = (k, 0) \in \mathcal{L}_n$, consider the derivations*

$$
\begin{aligned}
V &= x^{m-n} y^m \,\partial_{\mathrm{z}} \\
X &= x \,\partial_{\mathrm{x}} + (1 + x^k y^k)(-y \,\partial_{\mathrm{y}} - nz \,\partial_{\mathrm{z}}) \text{ and} \\
Y &= X + (m - n) x^{k+m-n} y^{m+k} \,\partial_{\mathrm{z}} .
\end{aligned}
$$

By proposition C, the crossing type foliations generated by X and Y have conjugated holonomies but are not orbitally analytically equivalent.

5.6 Crossing type foliations in $\mathbb{C}^4$

In this section, we study crossing type foliations in $(\mathbb{C}^4, 0)$ whose local generators have real linear parts. For such foliations, the extra hypothesis over the eigenvalues together with the no transverse negative resonance property allow us to formulate the following.

Corollary 5.6.1. *Consider a crossing type foliation $(\mathcal{F}, H, \Gamma) \in (\mathbb{C}^4, 0)$ whose local generator X has a real linear part with μ_1, μ_2, μ_3, and μ_4 as the eigenvalues such that $\mu_1^{-1} X$ is an x-normalized derivation. Then tree cases can appear:*

1. *The eigenvalues of X are in the Poincaré domain. Then $(\mathcal{F}, H, \Gamma)$ is analytically normalizable and has at most a finite number of resonant monomials.*

2. *The eigenvalues of X are in the Siegel domain. Then $(\mathcal{F}, H, \Gamma)$ is analytically classified by its linear part and its Γ-holonomy.*

3. *The eigenvalues of X are in the Siegel domain and*

   (a) *$\mu_i \in \mathbb{C} \setminus \mathbb{R}$ is non-pure imaginary for $i = 1, 2, 3, 4$, $\mu_1 = \overline{\mu_2}$, $\mu_3 = \overline{\mu_4}$, and up to an axis permutation*

   $$\begin{cases} \mu_4 \in \mathbb{Z}_{\geq 1} \,\mu_2 + \mathbb{Z}_{\geq 0} \,\mu_3 - \mathbb{Z}_{\geq 1} \,\mu_1 \\ \mu_1 \notin \mathbb{Q}_{\geq 0} \,\mu_2 + \mathbb{Q}_{\geq 0} \,\mu_3 - \mathbb{Q}_{\geq 0} \,\mu_4 \end{cases} .$$

   (b) *$\mu_1 = \overline{\mu_2}$ are pure imaginary, $\mu_3 = \overline{\mu_4} \in \mathbb{C} \setminus \mathbb{R}$ are non-pure imaginary and*

   $$2|\mathrm{Im}(\mu_3)| \in \mathbb{Z}_{\geq 1} |\mathrm{Im}(\mu_1)|.$$

   (c) *μ_i is pure imaginary for $i = 1, 2, 3, 4$, and*

   $$|\mathrm{Im}(\mu_3)| \in \mathbb{Q}_{\geq 0} |\mathrm{Im}(\mu_1)|.$$

   (d) *Up to a rotation, $\mu_1, \mu_2 \in \mathbb{R}_{>0}$, $\mu_3, \mu_4 \in \mathbb{C} \setminus \mathbb{R}$, and $\mathrm{Re}(\mu_3) < 0$, and*

   $$\mu_2 \in \mathbb{Q}_{>0} \mu_1 - \mathbb{Q}_{>0} \,\mathrm{Re}(\mu_3).$$

(e) $\mu_1 > 0$*,* $\mu_2 < 0$*,* $\mu_3 = \overline{\mu_4} \in \mathbb{C} \setminus \mathbb{R}$*,* $\mathrm{Re}(\mu_3) > 0$*, and*

$$\begin{cases} \mu_2 \notin 2\mathbb{Z}_{\geq 1}|\mathrm{Re}(\mu_3)| - \mathbb{Z}_{\geq 1}\mu_1 \\ |\mathrm{Re}(\mu_3)| \notin \mathbb{Q}_{>0}\mu_1 \\ \mu_2 \notin \mathbb{Q}_{>0}\mu_1 - \mathbb{Q}_{>0}|\mathrm{Re}(\mu_3)| \end{cases}.$$

(f) $\mu_1 > 0$*,* $\mu_2 < 0$*,* $\mu_3 = \overline{\mu_4} \in \mathbb{C} \setminus \mathbb{R}$*,* $\mathrm{Re}(\mu_3) < 0$*, and*

$$\mu_2 \in 2\mathbb{Z}_{\geq 1}\,\mathrm{Re}(\mu_3) - \mathbb{Z}_{\geq 1}\mu_1.$$

(g) Up to a rotation, $\mu_1, \mu_2 > 0$*,* $\mu_3 = \overline{\mu_4}$ *are pure complex, and*

$$\mu_2 \notin \mathbb{Q}_{>0}\,\mu_1.$$

(h) $\mu_1 > 0$*,* $\mu_2 < 0$*,* $\mu_3 = \overline{\mu_4}$ *pure complex, and*

$$\mu_2 \notin \mathbb{Z}_{<0}\,\mu_1.$$

(i) $\mu_1 \in \mathbb{R}^*$*,* $\mu_2 = 0$*,* $\mu_3 = \overline{\mu_4} \in \mathbb{C} \setminus \mathbb{R}$*,* $\mathrm{Re}(\mu_3) > 0$*, and*

$$\mathrm{Re}(\mu_3) \in \mathbb{Q}_{>0}\mu_1.$$

(j) μ_i*'s are real such that we can construct Table 5.1*

Proof. For the Poincaré domain, the result is already known. Then, once for all, we assume that the eigenvalues of X are in the Siegel domain (so of $\mu_1^{-1}X$). Now recall that a derivation with eigenvalues $\mu_1 = 1, \mu_2, \mu_3$, and μ_4 does not satisfy the no transverse negative resonance property (or simply NTNR property) if exist $(p_2, p_3, p_4) \in \mathcal{L}_3$ and $q \in \mathbb{Z}_{\geq 1}$ satisfying

$$\sum_{i=2}^{4} p_i \mu_i = q. \tag{5.10}$$

Consider that all the eigenvalues of X are non real. Then, up to rotation, they have one of space distributions in $\mathbb{C}$ shown in Figure 5.4. For the left-hand side case, we can write the eigenvalues in the following form $\mu_1 = a + bi$, $\mu_2 = a - bi$, $\mu_3 = -c + di$, and $\mu_4 = -c - di$, where $a, b, c, d \in \mathbb{R}_{>0}$.

Suppose that $\mu_1^{-1}X$ does not satisfy the NTNR property. Hence equation (5.10) takes the form

$$\sum_{i=2}^{4} p_i \frac{\mu_i}{\mu_1} = q. \tag{5.11}$$

Then multiplying equation (5.11) by μ_1, we find

$$\sum_{i=2}^{4} p_i \mu_i = q\mu_1. \tag{5.12}$$

$\mu_1, \mu_2, \mu_3, \mu_4 > 0$	$\mu_1 \in \mathbb{Q}_{\geq -1}\, \mu_2 + \mathbb{Q}_{\geq 0}\, \mu_3 + \mathbb{Q}_{>0}\, \mu_4$
$\mu_1, \mu_2, \mu_3 > 0, \mu_4 < 0$	$\mu_4 \in \mathbb{Z}_{\geq 1}\, \mu_2 + \mathbb{Z}_{\geq 0}\, \mu_3 - \mathbb{Z}_{\geq 1}\, \mu_1$ $\mu_1 \in \mathbb{Q}_{\geq -1}\, \mu_2 + \mathbb{Q}_{\geq 0}\, \mu_3 + \mathbb{Q}_{>0}\, \mu_4$ $\mu_1 \in \mathbb{Q}_{\geq 0}\, \mu_2 + \mathbb{Q}_{\geq -1}\, \mu_3 + \mathbb{Q}_{\geq 0}\, \mu_4$
$\mu_1, \mu_2 > 0, \mu_3, \mu_4 < 0$	$\mu_1 \in \mathbb{Q}_{\geq -1}\, \mu_2 + \mathbb{Q}_{\geq 0}\, \mu_3 + \mathbb{Q}_{>0}\, \mu_4$ $\mu_1 \in \mathbb{Q}_{\geq 0}\, \mu_2 + \mathbb{Q}_{\geq -1}\, \mu_3 + \mathbb{Q}_{>0}\, \mu_4$ $\mu_1 \in \mathbb{Q}_{>0}\, \mu_2 + \mathbb{Q}_{\geq -1}\, \mu_3 + \mathbb{Q}_{\geq 0}\, \mu_4$
$\mu_1 > 0, \mu_2, \mu_3, \mu_4 < 0$	$\mu_1 \in -\mu_2 + \mathbb{Q}_{\geq 0}\, \mu_3 + \mathbb{Q}_{>0}\, \mu_4$
$\mu_1, \mu_2, \mu_3, \mu_4 < 0$	$\mu_1 \in \mathbb{Q}_{\geq -1}\, \mu_2 + \mathbb{Q}_{\geq 0}\, \mu_3 + \mathbb{Q}_{>0}\, \mu_4$
$\mu_1, \mu_2, \mu_3 > 0, \mu_4 = 0$	$\mu_2 \in \mathbb{Z}_{\geq 1}\, \mu_3 - \mathbb{Z}_{\geq 1}\, \mu_1$ $\mu_1 \in \mathbb{Q}_{>0}\, \mu_2 + \mathbb{Q}_{>0}\, \mu_3$
$\mu_1, \mu_2 > 0, \mu_3 < 0, \mu_4 = 0$	$\mu_3 \in \mathbb{Z}_{\geq 0}\, \mu_2 - \mathbb{Z}_{\geq 1}\, \mu_1$ $\mu_1 \in \mathbb{Q}_{\geq 0}\, \mu_2 - \mathbb{Q}_{\geq 0}\, \mu_3$
$\mu_1 > 0, \mu_2, \mu_3 < 0, \mu_4 = 0$	$\mu_2 \in \mathbb{Z}_{\geq 1}\, \mu_3 - \mathbb{Z}_{\geq 1}\, \mu_1$
$\mu_1, \mu_2, \mu_3 < 0, \mu_4 = 0$	$\mu_2 \in \mathbb{Z}_{\geq 0}\, \mu_3 - \mathbb{Z}_{\geq 1}\, \mu_1$
$\mu_1, \mu_2 > 0, \mu_3, \mu_4 = 0$	$\mu_2 \in \mathbb{Q}_{>0}\, \mu_1$
$\mu_1 > 0, \mu_2 < 0, \mu_3, \mu_4 = 0$	$\mu_2 \in -\mathbb{Z}_{>1}\, \mu_1$
$\mu_1, \mu_2 < 0, \mu_3, \mu_4 = 0$	$\mu_2 \in \mathbb{Q}_{>0}\, \mu_1$
$\mu_1 > 0, \mu_2, \mu_3, \mu_4 = 0$	Always satisfy
$\mu_1 < 0, \mu_2, \mu_3, \mu_4 = 0$	Always satisfy
$\mu_1, \mu_2, \mu_3, \mu_4 = 0$	Always satisfy

Table 5.1: Table for real eigenvalues

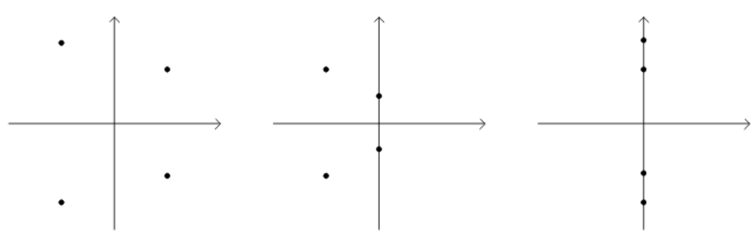


Figure 5.4: Space distribution for four non real eigenvalues in Siegel domain.

If $p_4 = -1, p_3 = 0, p_2 \geq 1$, Equation (5.12) implies that

$$\mu_4 = \mathbb{Z}_{\geq 1}\, \mu_2 - \mathbb{Z}_{\geq 1}\, \mu_1.$$

Therefore, if $\mu_4 \notin \mathbb{Z}_{\geq 1}\mu_2 - \mathbb{Z}_{\geq 1}\mu_1$, Equation (5.12) is never satisfied. Analogously, we can construct the Table 5.2.

Hypotheses	Restrictions
$p_4 = -1, p_3 = 0, p_2 \geq 1$	$\mu_4 \notin \mathbb{Z}_{\geq 1}\,\mu_2 - \mathbb{Z}_{\geq 1}\,\mu_1$
$p_4 = -1, p_3 \geq 1, p_2 \geq 1$	$\mu_4 \notin \mathbb{Z}_{\geq 1}\,\mu_2 + \mathbb{Z}_{\geq 1}\,\mu_3 - \mathbb{Z}_{\geq 1}\,\mu_1$
$p_4 = 0, p_3 \geq 1, p_2 \geq 1$	$\mu_1 \notin \mathbb{Q}_{\geq 0}\,\mu_2 + \mathbb{Q}_{\geq 0}\,\mu_3$
$p_4 \geq 1, p_3 = -1, p_2 \geq 1$	$\mu_3 \notin \mathbb{Z}_{\geq 1}\,\mu_2 + \mathbb{Z}_{\geq 0}\,\mu_4 - \mathbb{Z}_{\geq 1}\,\mu_1$
$p_4 \geq 1, p_3 \geq 1, p_2 \geq 1$	$\mu_1 \notin \mathbb{Q}_{\geq 0}\,\mu_2 + \mathbb{Q}_{\geq 0}\,\mu_3 - \mathbb{Q}_{\geq 0}\,\mu_4$

Table 5.2: Restriction for the first non real case.

On the other hand, isolating the real and imaginary parts in (5.12), we find

$$\begin{aligned}(p_3 + p_4)c &= (p_2 - q)a, \\ (p_3 - p_4)d &= (p_2 + q)b.\end{aligned} \tag{5.13}$$

Apart from the possibilities shown in Table 5.2, the contradictions that System (5.13) leads are simple to infer. However, in the seek of completeness, Table 5.3 shows for each assumption over p_2, p_3, and p_4 the correspondent contradiction.

Hypotheses	Contradiction leaded by equations (5.13)
$p_4 = -1, p_3 \geq 1, p_2 = 0$	$(p_3 - 1)c = -qa$
$p_4 = 0, p_3 = -1, p_2 \geq 1$	$-p_3 d = (p_2 + q)b$
$p_4 = 0, p_3 = 0, p_2 = 0$	$0 = q\mu_1$
$p_4 = 0, p_3 = 0, p_2 \geq 1$	$0 = (p_2 + q)b$
$p_4 = 0, p_3 \geq 1, p_2 = -1$	$p_3 c = (-1 - q)a$
$p_4 = 0, p_3 \geq 1, p_2 = 0$	$p_3 c = -qa$
$p_4 \geq 1, p_3 = -1, p_2 = 0$	$p_4 c = -qa$
$p_4 \geq 1, p_3 = 0, p_2 \geq 1$	$p_4 c = (-1 - q)a$
$p_4 \geq 1, p_3 = 0, p_2 \geq 1$	$-p_4 d = (p_2 + q)b$
$p_4 \geq 1, p_3 \geq 1, p_2 = -1$	$(p_3 + p_4)c = (-1 - q)a$
$p_4 \geq 1, p_3 \geq 1, p_2 = 0$	$(p_3 + p_4)c = -qa$

Table 5.3: Contradictions for first non real case.

With respect to the center distribution in Figure 5.4, we can write $\mu_1 = bi$, $\mu_2 = -bi$, $\mu_3 = -c + di$, and $\mu_4 = -c - di$, where $b, c, d \in \mathbb{R}_{>0}$. Then the System (5.13) takes the following form

$$-(p_3 + p_4)c = 0, \tag{5.14}$$

$$-p_2 b + (p_3 - p_4)d = qb. \tag{5.15}$$

By (5.14), we infer that $p_3 = -p_4$, and consequently $p_4 \in \{-1, 0, 1\}$. If $p_4 = p_3 = 0$ and $p_2 \geq 0$ or $p_4 = 1 = -p_3$ and $p_2 \geq 0$, by Equation (5.15), we have that $0 \geq -p_2 b + (p_3 - p_4)d = qb$, which is a contradiction. In the remaining case, if $2\,\mathrm{Im}(\mu_3) \notin \mathbb{Z}_{\geq 1}\,\mathrm{Im}(\mu_1)$, or equivalently $2|\mathrm{Im}(\mu_3)| \notin \mathbb{Z}_{\geq 1}|\mathrm{Im}(\mu_1)|$, the NTNR property is satisfied.

Hypotheses	Restrictions
$p_4 = -1, p_3 = 0, p_2 \geq 1$	$\lvert\text{Im}(\mu_3)\rvert \notin \mathbb{Z}_{\geq 2}\lvert\text{Im}(\mu_1)\rvert$
$p_4 = -1, p_3 \geq 1, p_2 = 0$	$\lvert\text{Im}(\mu_3)\rvert \notin \mathbb{Q}_{\geq 0}\lvert\text{Im}(\mu_1)\rvert$
$p_4 = -1, p_3 \geq 1, p_2 \geq 1$	$\lvert\text{Im}(\mu_3)\rvert \notin \mathbb{Q}_{\geq 0}\lvert\text{Im}(\mu_1)\rvert$
$p_4 = 0, p_3 \geq 1, p_2 = -1$	$\lvert\text{Im}(\mu_3)\rvert \notin \mathbb{Q}_{\geq 0}\lvert\text{Im}(\mu_1)\rvert$
$p_4 = 0, p_3 \geq 1, p_2 = 0$	$\lvert\text{Im}(\mu_3)\rvert \notin \mathbb{Q}_{>0}\lvert\text{Im}(\mu_1)\rvert$
$p_4 = 0, p_3 \geq 1, p_2 \geq 1$	$\lvert\text{Im}(\mu_3)\rvert \notin \mathbb{Q}_{>0}\lvert\text{Im}(\mu_1)\rvert$
$p_4 \geq 1, p_3 \geq 1, p_2 = 0$	$\lvert\text{Im}(\mu_3)\rvert \notin \mathbb{Q}_{\geq 0}\lvert\text{Im}(\mu_1)\rvert$
$p_4 \geq 1, p_3 \geq 1, p_2 = -1$	$\lvert\text{Im}(\mu_3)\rvert \notin \mathbb{Q}_{>0}\lvert\text{Im}(\mu_1)\rvert$
$p_4 \geq 1, p_3 \geq 1, p_2 \geq 1$	$\lvert\text{Im}(\mu_3)\rvert \notin \mathbb{Q}_{>0}\lvert\text{Im}(\mu_1)\rvert$

Table 5.4: Restriction for the second non real case.

Hypotheses	Contradiction leaded by Equation (5.16)
$p_4 = 0, p_3 = -1, p_2 \geq 1$	$-d = (q + p_2)b$
$p_4 = 0, p_3 = 0, p_2 = 0$	$0 = qb$
$p_4 = 0, p_3 = 0, p_2 \geq 1$	$0 = (q + p_2)b$
$p_4 \geq 1, p_3 = -1, p_2 = 0$	$(-1 - p_4)d = qb$
$p_4 \geq 1, p_3 = -1, p_2 \geq 1$	$(-1 - p_4)d = (q + p_2)b$
$p_4 \geq 1, p_3 = 0, p_2 = -1$	$-p_4 d = (q - 1)b$
$p_4 \geq 1, p_3 = 0, p_2 = 0$	$-p_4 d = qb$
$p_4 \geq 1, p_3 = 0, p_2 \geq 1$	$-p_4 d = (q + p_2)b$

Table 5.5: Restriction for the third non real case.

Finally, for the right-hand side distribution in Figure 5.4, we can assume that $\mu_1 = ib$, $\mu_2 = -ib$, $\mu_3 = id$, and $\mu_4 = -id$. Then Equation (5.13) becomes

$$(p_3 - p_4)d = (q + p_2)b. \tag{5.16}$$

If $p_3 = p_4$, we have that $q = -p_2$, which is a contradiction. On the other hand for $p_3 \neq p_4$, we can construct the Table 5.4 using Equation (5.16). In particular, if $|\text{Im}(\mu_3)| \notin \mathbb{Q}_{\geq 0}|\text{Im}(\mu_1)|$, the NTNR property holds. In seek of completeness, we show Table 5.5 that contains the contradictions that Equation (5.16) leads.

Now, assume that X has two non-null real and two non-real and non-pure imaginary eigenvalues. Accordingly, we have, up to rotation, three possible space distributions as shown in Figure 5.5. To the distributions in the right hand side, we can assume $\mu_1, \mu_2 > 0$, $\mu_3 = \overline{\mu_4} = -c + di$, where $c, d \in \mathbb{R}_{>0}$. Hence Equation (5.13) becomes

$$p_2\mu_2 - (p_3 + p_4)c = q\mu_1, \tag{5.17}$$

$$(p_3 - p_4)d = 0. \tag{5.18}$$

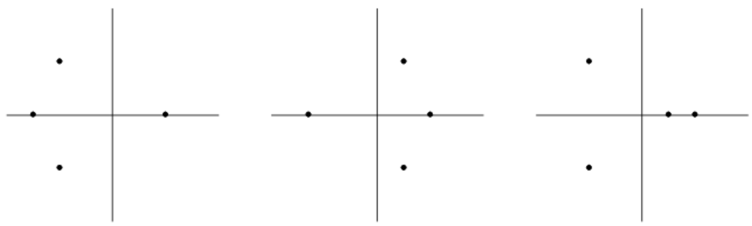


Figure 5.5: Real and non-real eigenvalues space distribution.

Equation (5.18) implies that $p_3 = p_4$. And if $p_2 \in \{-1, 0\}$, we have $0 > p_2\mu_2 - (p_3 + p_4)c = q\mu_1$, which is a contradiction. On the other hand, if $p_2 > 0$ and $\mu_2 \notin \mathbb{Q}_{>0}\mu_1 - \mathbb{Q}_{>0}\operatorname{Re}(\mu_3)$, the NTNR property is satisfied.

Now, for the distribution in the center, we can assume $\mu_1 > 0$, $\mu_2 < 0$, and $\mu_3 = \overline{\mu_4} = c + di$, where $c, d \in \mathbb{R}_{>0}$. From Equation (5.13), we have that

$$p_2\mu_2 + (p_3 + p_4)c = q\mu_1, \tag{5.19}$$
$$(p_3 - p_4)d = 0. \tag{5.20}$$

By Equation (5.20), we find $p_3 = p_4$. In particular, for $p_4 = p_3 = p_2 = 0$ and $p_4 = p_3 = 0$ and $p_2 = 1$, we have that $0 \geq p_2\mu_2 = q$, which is a contradiction. Moreover, by a simple analysis of the Equations (5.19) and (5.20), we can construct Table 5.6.

Hypothesis	Restrictions
$p_4 \geq 1, p_3 \geq 1, p_2 = -1$	$\mu_2 \notin 2\mathbb{Z}_{\geq 1}\lvert\operatorname{Re}(\mu_3)\rvert - \mathbb{Z}_{\geq 1}\mu_1$
$p_4 \geq 1, p_3 \geq 1, p_2 = 0$	$\lvert\operatorname{Re}(\mu_3)\rvert \notin \mathbb{Q}_{>0}\mu_1$
$p_4 \geq 1, p_3 \geq 1, p_2 \geq 1$	$\mu_2 \notin \mathbb{Q}_{>0}\mu_1 - \mathbb{Q}_{>0}\lvert\operatorname{Re}(\mu_3)\rvert$

Table 5.6: Restrictions for the top left case.

Now, we analyse the left-hand case, where we can write $\mu_1 > 0$, $\mu_2 < 0$ and $\mu_3 = \overline{\mu_4} = -c + di$, with $c, d \in \mathbb{R}_{>0}$. As in the previous cases, we have

$$p_2\mu_2 - (p_3 + p_4)c = q\mu_1,$$
$$(p_3 - p_4)d = 0,$$

and consequently $p_3 = p_4$. In addition, if $p_2 \geq 0$, we have $0 \geq p_2\mu_2 - (p_3 + p_4)c = q\mu_1$, which is a contradiction. Moreover, if $p_2 = -1$ and $\mu_2 \notin -2\mathbb{Z}_{\geq 1}\operatorname{Re}(\mu_3) - \mathbb{Z}_{\geq 1}\mu_1$, it is simple to see that X satisfies the NTNR property.

For the case where the eigenvalues are two non null real numbers and two pure complex numbers, we have, up to rotation, two possible space distribution as shown in Figure 5.6.

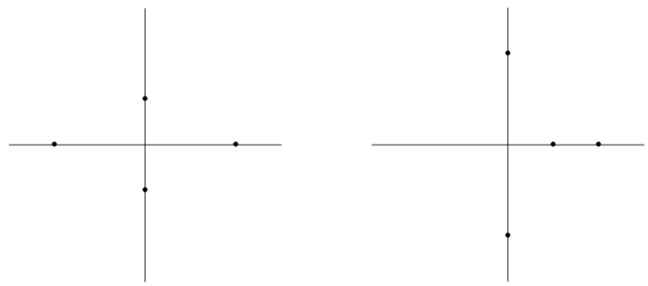


Figure 5.6: Real non null and pure complex eigenvalues space distribution.

For the right-hand side, we can write $\mu_1, \mu_2 > 0$, $\mu_3 = di = \overline{\mu_4}$, where $d \in \mathbb{R}_{>0}$. Then Equation (5.13) takes the form:

$$\begin{aligned} p_2\mu_2 &= q, \\ (p_3 - p_4)bi &= 0. \end{aligned}$$

It implies that $p_3 = p_4 \geq 0$. Therefore, if $\mu_2 \notin \mathbb{Q}_{>0}\,\mu_1$, the NTNR property is satisfied. On the other hand, for the space distribution in the left-hand side, we can write $\mu_1 > 0$, $\mu_2 < 0$, $\mu_3 = di = \overline{\mu_4}$, where $d \in \mathbb{R}_{>0}$. Analogous, we find that if $\mu_2 \notin -\mathbb{Z}_{>0}\,\mu_1$, the NTNR property is satisfied.

Now, let $\mu_1 \in \mathbb{R}^*$, $\mu_2 = 0$, and $\mu_3 = \overline{\mu_4} = c + di$, where $c \in \mathbb{R}$ and $d \in \mathbb{R}^*$. Equation (5.13) becomes

$$\begin{aligned} (p_3 + p_4)c &= q\mu_1, \\ (p_3 - p_4)d &= 0. \end{aligned}$$

Hence $p_3 = p_4$, and consequently, if $\mathrm{Re}(\mu_3) \notin \mathbb{Q}_{>0}\mu_1$, the NTNR property is satisfied.

For the last case with complex eigenvalues, we can write $\mu_1 = \overline{\mu_2} = a + bi$, where $a \in \mathbb{R}_{>0}, b \in \mathbb{R}_{>0}$, and $\mu_3 = \mu_4 = 0$. Equation (5.13) becomes

$$\begin{aligned} p_2\, a &= qa, \\ -\, p_2\, b &= qb. \end{aligned}$$

Therefore, $p_2 = q = -p_2$, and consequently $p_2 = 0$ and $q = 0$, which is a contradiction.

In the case where all eigenvalues are real, we have 14 sub-cases accordingly to their position in the real line. In any of these cases, the analysis is made as previously: We assume that the eigenvalues do not satisfy the NTNR property and in the sequel we analyse Equations (5.12) and (5.13) with different values of p_2, p_3, and p_4. We give Table 5.7 with the respective set where the NTNR is valid. $\square$

Hypothesis	Restrictions to satisfy the NTNR property
$\mu_1, \mu_2, \mu_3, \mu_4 > 0$	$\mu_1 \notin \mathbb{Q}_{\geq -1}\, \mu_2 + \mathbb{Q}_{\geq 0}\, \mu_3 + \mathbb{Q}_{>0}\, \mu_4$
$\mu_1, \mu_2, \mu_3 > 0, \mu_4 < 0$	$\mu_4 \notin \mathbb{Z}_{\geq 1}\, \mu_2 + \mathbb{Z}_{\geq 0}\, \mu_3 - \mathbb{Z}_{\geq 1}\, \mu_1$ $\mu_1 \notin \mathbb{Q}_{\geq -1}\, \mu_2 + \mathbb{Q}_{\geq 0}\, \mu_3 + \mathbb{Q}_{>0}\, \mu_4$ $\mu_1 \notin \mathbb{Q}_{\geq 0}\, \mu_2 + \mathbb{Q}_{\geq -1}\, \mu_3 + \mathbb{Q}_{\geq 0}\, \mu_4$
$\mu_1, \mu_2 > 0, \mu_3, \mu_4 < 0$	$\mu_1 \notin \mathbb{Q}_{\geq -1}\, \mu_2 + \mathbb{Q}_{\geq 0}\, \mu_3 + \mathbb{Q}_{>0}\, \mu_4$ $\mu_1 \notin \mathbb{Q}_{\geq 0}\, \mu_2 + \mathbb{Q}_{\geq -1}\, \mu_3 + \mathbb{Q}_{>0}\, \mu_4$ $\mu_1 \notin \mathbb{Q}_{>0}\, \mu_2 + \mathbb{Q}_{\geq -1}\, \mu_3 + \mathbb{Q}_{\geq 0}\, \mu_4$
$\mu_1 > 0, \mu_2, \mu_3, \mu_4 < 0$	$\mu_1 \notin -\mu_2 + \mathbb{Q}_{\geq 0}\, \mu_3 + \mathbb{Q}_{>0}\, \mu_4$
$\mu_1, \mu_2, \mu_3, \mu_4 < 0$	$\mu_1 \notin \mathbb{Q}_{\geq -1}\, \mu_2 + \mathbb{Q}_{\geq 0}\, \mu_3 + \mathbb{Q}_{>0}\, \mu_4$
$\mu_1, \mu_2, \mu_3 > 0, \mu_4 = 0$	$\mu_2 \notin \mathbb{Z}_{\geq 1}\, \mu_3 - \mathbb{Z}_{\geq 1}\, \mu_1$ $\mu_1 \notin \mathbb{Q}_{>0}\, \mu_2 + \mathbb{Q}_{>0}\, \mu_3$
$\mu_1, \mu_2 > 0, \mu_3 < 0, \mu_4 = 0$	$\mu_3 \notin \mathbb{Z}_{\geq 0}\, \mu_2 - \mathbb{Z}_{\geq 1}\, \mu_1$ $\mu_1 \notin \mathbb{Q}_{\geq 0}\, \mu_2 - \mathbb{Q}_{\geq 0}\, \mu_3$
$\mu_1 > 0, \mu_2, \mu_3 < 0, \mu_4 = 0$	$\mu_2 \notin \mathbb{Z}_{\geq 1}\, \mu_3 - \mathbb{Z}_{\geq 1}\, \mu_1$
$\mu_1, \mu_2, \mu_3 < 0, \mu_4 = 0$	$\mu_2 \notin \mathbb{Z}_{\geq 0}\, \mu_3 - \mathbb{Z}_{\geq 1}\, \mu_1$
$\mu_1, \mu_2 > 0, \mu_3, \mu_4 = 0$	$\mu_2 \notin \mathbb{Q}_{>0}\, \mu_1$
$\mu_1 > 0, \mu_2 < 0, \mu_3, \mu_4 = 0$	$\mu_2 \notin -\mathbb{Z}_{>1}\, \mu_1$
$\mu_1, \mu_2 < 0, \mu_3, \mu_4 = 0$	$\mu_2 \notin \mathbb{Q}_{>0}\, \mu_1$
$\mu_1 > 0, \mu_2, \mu_3, \mu_4 = 0$	Always satisfy
$\mu_1 < 0, \mu_2, \mu_3, \mu_4 = 0$	Always satisfy
$\mu_1, \mu_2, \mu_3, \mu_4 = 0$	Always satisfy

Table 5.7: Table for real eigenvalues and its restrictions

Introduction en français

La classification analytique des feuilletages singuliers en dimension deux et sa connexion avec la conjugaison analytique des holonomies était un des résultats centraux de l'article de Mattei et Moussu [MM80b] en 1980.

Plus tard en 1984, Elizarov et Il'Yashenko [EI84] ont prouvé que, en dimension trois ou plus, si nous ajoutons quelques restrictions sur les champs vectoriels qui génèrent les feuilletages, la conjugaison analytique des holonomies correspond à l'équivalence analytique des feuilletages. En 2006, Ce même résultat fut prouvé en 2006 par Helena Reis [Rei06] avec une méthode plus simple.

Plus précisément, les auteurs considèrent des germes de champs vectoriels analytiques singuliers X dans $(\mathbb{C}^n, \mathbf{0})$ pour $n \geq 3$, avec $\lambda_1, \ldots, \lambda_n$ comme valeurs propres de la partie linéaire de X, vérifiant:

1. L'origine de $\mathbb{C}^n$ est une singularité isolée de X.

2. X est de type Siegel (c'est-à-dire que l'enveloppe convexe de $\lambda_1, \ldots, \lambda_n$ contient l'origine).

3. Toutes les valeurs propres sont non nulles et il existe une droite passant par l'origine de $\mathbb{C}$ séparant λ_1 des autres valeurs propres dans le plan complexe.

4. À un changement de coordonnées près, $X = \sum_{i=1}^{n} \lambda_i z_i (1 + f_i(z)) \partial_{z_i}$, où $z = (z_1, \ldots, z_n)$, et f_i est un germe de fonction analytique tel que $f_i(0) = 0$ pour tout i.

Dans [EI84] et [Rei06], il est prouvé ce qui suit :

Théorème ([Rei06], Théorème 1)**.** *Soient X et Y deux germes des champs vectoriels, vérifiant* $(1), (2), (3)$ *et* (4)*. Notons Δ_X et Δ_Y les holonomies de X et Y relativement aux séparatrices de X et Y tangentes à l'espace propre associé à la première valeur propre, respectivement. Alors si Δ_X et Δ_Y sont analytiquement conjugués, donc X et Y sont analytiquement équivalents.*

Dans ce travail, nous enlevons l'hypothèse (1) et affaiblissons $(2), (3)$ et (4). En conséquence, nous élargissons l'ensemble des champs vectoriels pour lesquels la conclusion du théorème est vraie.

Plus précisément, nous traitons une classe de germes de feuilletages analytiques singuliers appelés de type croisement. Un feuilletage de *type croisement* dans $(\mathbb{C}^{n+1}, 0)$ est un triplet $(\mathcal{F}, H, \Gamma)$ tel que:

i. $\mathcal{F}$ est un germe de feuilletage analytique à une dimension.

ii. H est une hyper-surface lisse et Γ est une courbe invariante lisse telle que:

  (a) H et Γ sont transverses à l'origine.

  (b) Les deux sont invariants par le feuilletage $\mathcal{F}$.

iii. Chaque générateur local de $\mathcal{F}$ a une valeur propre non nulle dans la direction Γ.

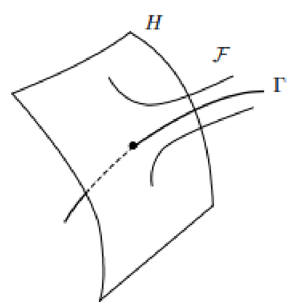


Figure 5.7: Transversalité de H et Γ à l'origine

Comme dans les articles cités ci-dessus, nous devons exiger une propriété sur les valeurs propres de la partie linéaire des générateurs locaux. On dit qu'un champ de vecteurs, avec $1, \mu_1 \ldots, \mu_n$ comme valeurs propres de sa partie linéaire, n'a *pas de résonance transverse négative* si aucun élément dans le cône positif $\mathcal{C} = \{\sum_{i=1}^{n} p_i\mu_i; p_1 + \cdots + p_n \geq 1\}$, où $p_i \in \mathbb{Z}_{\geq 0}$, peut être écrit sous la forme $\mu_j + q$, avec $q \in \mathbb{Z}_{\geq 1}$, pour tout $1 \leq j \leq n$.

Comme conséquence de la définition des feuilletages de type croisement, il existe des coordonnées locales $(x, \mathbf{z})$, dites adaptées à $(\mathcal{F}, H, \Gamma)$, telles que la courbe Γ et l'hypersurface H sont exprimés respectivement par $\gamma := \{\mathbf{z} = 0\}$, $H := \{x = 0\}$. De plus, si les générateurs locaux de $(\mathcal{F}, H, \Gamma)$ n'ont pas de résonance transverse négative, on peut choisir un générateur local dans ces coordonnées adaptées qui a la forme

$$x\partial_x + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} z_j \partial_{z_i} + \sum_{i=1}^{n} b_i(x, \mathbf{z})\partial_{z_i}, \tag{5.21}$$

où $(a_{ij})_{n\times n}$ est une matrice constante, et $b_i(x, 0) = \frac{\partial b_i}{\partial z_j}(x, 0) = 0$ pour tout $i, j \in \{1, \ldots, n\}$. On dit qu'un champ de vecteurs avec cette forme est un champ de vecteurs *x-normalisé*.

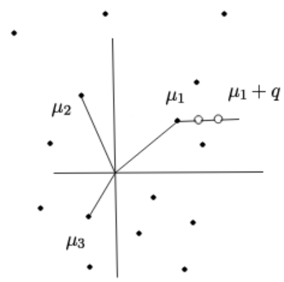


Figure 5.8: Pas de résonance négative transversale

Notre objectif principal est de classer ces feuilletages analytiques singuliers jusqu'à l'équivalence analytique. Ici, on dit que deux feuilletages de type croisement $(\mathcal{F}, H, \Gamma)$ et $(\mathcal{G}, L, \Omega)$ sont *analytiquement équivalents* s'il y a un changement de coordonnées analytique qui envoie les feuilles de $\mathcal{F}$ aux feuilles de $\mathcal{G}$ et la paire (H, Γ) à (L, Ω).

Le théorème suivant est le résultat principal de ce travail. Nous avons adapté et généralisé une idée initialement introduite dans la thèse d'Arame Diaw [Dia19, DL20] pour le prouver. Ci-dessous, nous notons par Γ-*holonomy* et Ω-*holonomy* les holonomies locales respectives le long des courbes Γ de $\mathcal{F}$ et Ω de $\mathcal{G}$.

Théorème Principal. *Soient $(\mathcal{F}, H, \Gamma)$ et $(\mathcal{G}, L, \Omega)$ deux feuilletages de type croisement tels que :*

(a) Les parties linéaires des générateurs locaux de $\mathcal{F}$ et $\mathcal{G}$ sont conjuguées.

(b) Le générateur local de $\mathcal{F}$ (et donc celui de $\mathcal{G}$) n'a pas de résonance transverse négative.

(c) Les Γ-holonomie et Ω-holonomie sont analytiquement conjuguées.

Alors $(\mathcal{F}, H, \Gamma)$ et $(\mathcal{G}, L, \Omega)$ sont analytiquement équivalents.

Motivés par le Théorème Principal ci-dessus, nous disons qu'un feuilletage de type croisement $(\mathcal{F}, H, \Gamma)$ est *classé analytiquement par sa partie linéaire et son* Γ-*holonomie* si tous les feuilletages de type croisement $(\mathcal{G}, L, \Omega)$ avec une partie linéaire conjuguée et une Ω-holonomie conjuguée à $(\mathcal{F}, H, \Gamma)$ est analytiquement équivalent à $(\mathcal{F}, H, \Gamma)$.

En conséquence, nous pouvons donner une preuve unifiée d'un résultat obtenu avec différents outils par Mattei et Moussu [MM80b] et plus tard par Martinet et Ramis [MR82]. Dans un article récent [DL20], Diaw et Loray utilisent des techniques similaires pour prouver ce théorème. Avec notre notation, nous pouvons l'énoncer comme suit :

Corollaire A. *Soit un feuilletage de type croisement* $(\mathcal{F}, H, \Gamma) \in (\mathbb{C}^2, 0)$ *qui possède, en coordonnées adaptées, un générateur local* x*-normalisé de la forme*

$$X = x\partial_x + yf(x, y)\partial_y,$$

où f *est un germe de fonction analytique telle que* $f(0, 0) = \lambda$*. Deux cas sont possibles :*

1. *La valeur propre* λ *appartient à* $\mathbb{C}\backslash\mathbb{R}_{\leq 0}$*, alors* $(\mathcal{F}, H, \Gamma)$ *est analytiquement linéarisable.*
2. *La valeur propre* λ *appartient à* $\mathbb{R}_{\leq 0}$*, alors* $(\mathcal{F}, H, \Gamma)$ *est classé analytiquement par sa partie linéaire et son* Γ*-holonomie.*

On rappelle qu'un germe de champ de vecteurs singuliers dans $(\mathbb{C}^n, \mathbf{0})$ avec des valeurs propres $\lambda_1, \ldots, \lambda_n$ est dans le domaine de Siegel (resp. Poincaré) si l'origine est (resp. n'est pas) dans l'enveloppe convexe des valeurs propres en $\mathbb{C}$.

Proof. Dans le cas de Poincaré ($\lambda \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$), le résultat est immédiat puisque l'existence de deux séparatrices analytiques implique qu'il ne peut y avoir de résonance de type Poincaré. Dans le cas de Siegel ($\lambda \in \mathbb{R}_{\leq 0}$), il suffit de remarquer que la condition de pas de résonance transverse négative est satisfaite. Le résultat est donc une conséquence du Théorème Principal. □

En dimension trois, le corollaire suivant, qui sera démontré dans la section 5.3, est une généralisation de ce résultat.

Corollaire B. *Soit un feuilletage de type croisement* $(\mathcal{F}, H, \Gamma) \in (\mathbb{C}^3, 0)$ *qui a, en coordonnées adaptées, un générateur local* x*-normalisé avec partie semi-simple*

$$x\partial_x + \lambda y\partial_y + \mu z\partial_z.$$

Alors trois cas sont possibles :

1. *Les valeurs propres sont dans le domaine de Poincaré. Alors* $(\mathcal{F}, H, \Gamma)$ *est analytiquement normalisable et possède au maximum un nombre fini de monômes résonnants.*
2. *Les valeurs propres sont dans le domaine de Siegel et au moins une des valeurs propres* λ, μ *est non réelle. Alors* $(\mathcal{F}, H, \Gamma)$ *est classé analytiquement par sa partie linéaire et son* Γ*-holonomie.*
3. *Les valeurs propres sont dans le domaine de Siegel et toutes réelles. Alors* $(\mathcal{F}, H, \Gamma)$ *est classé analytiquement par sa partie linéaire et son holonomie ou une des conditions suivantes est vérifiée, à une permutation des coordonnées* y *et* z *près :*

(a) *Soit* $\mu < \lambda \leq 0$ *et*

$$p\lambda = \mu + q, \tag{5.22}$$

pour certains $p, q \in \mathbb{Z}_{\geq 1}$,

(b) *Ou,* $\mu \leq 0 < \lambda$*, satisfaisant* (5.22) *ou* $\lambda \in \mathbb{Q}_{>0} - \mu\mathbb{Q}_{\geq 0}$ *(notez que ces conditions ne sont pas mutuellement exclusives).*

En dimensions supérieures, une liste similaire de cas possibles est considérablement plus compliquée. Dans la dernière section, nous allons prouver un résultat similaire en dimension 4, avec l'hypothèse supplémentaire que la partie linéaire du générateur local est une matrice réelle. L'énoncé est assez technique et nous nous référons à la section 5.6 pour les détails.

La nécessité de la condition de non résonance transverse négative dans le Théorème Principal est une question naturelle. Dans ce sens, la Proposition C ci-dessous donne une recette pour construire des exemples où cette condition est mise en défaut. En particulier, nous obtenons des feuilletages de type croisement non équivalents avec des holonomies conjuguées.

Soit un champ vectoriel diagonal $S = x\,\partial_{\mathrm{x}} + L(\mu)$, où $L(\mu) = \sum_{i=1}^{n} \mu_i z_i\,\partial_{\mathrm{z_i}}$, $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$. Les *ensembles résonnants négatifs et positifs* de S sont donnés par

$$\mathrm{NR}(S) := \{T \in \mathcal{L}_n \,; \langle \mu, T\rangle \in \mathbb{Z}_{\geq 1}\},$$
$$\mathrm{PR}(S) := \{K \in \mathcal{L}_n \setminus \{\mathbf{0}\} \,; \langle \mu, K\rangle \in \mathbb{Z}_{\leq 0}\},$$

où $\langle \cdot, \cdot \rangle$ est le produit scalaire habituel du $\mathbb{C}^n$, et nous notons par $\mathcal{L}_n$ de n-uples $K \in \{(\mathbb{N}^n - e_1) \cup \cdots \cup (\mathbb{N}^n - e_n)\}$, tel que $|K| \geq 0$.

Le résultat suivant sera démontré dans la Section 5.5.

Proposition C. *Soit* $\mu = (\mu_1, \ldots, \mu_n)$ *un n-uplet dans* $\mathbb{C}^n$ *tel que* $\mu_i \neq 1$ *et* $\mu_i \neq \mu_j$*. Supposons qu'il existe un couple* $(T, K) \in \mathrm{NR}(S) \times \mathrm{PR}(S)$ *et un vecteur non nul* $\lambda \in \mathbb{C}^n$ *tel que*

1. $\langle \mu, T\rangle + \langle \mu, K\rangle \leq 0$,
2. *le champ vectoriel* $\mathbf{z}^T L(\lambda)$ *est analytique, et*
3. $\langle \lambda, K\rangle = 0$,

alors il existe deux champs vectoriels analytiques x-normalisés X, Y*, avec comme parties semi-simples* S*, qui génèrent des feuilletages analytiques qui ont des holonomies conjuguées le long de l'axe x, mais qui ne sont pas orbitalement analytiquement équivalents.*

Comme application, nous obtenons l'exemple suivant en dimension 3.

Example 1. *Les champs vectoriels*

$$X = x\,\partial_{\mathrm{x}} + 2(1 + x^2 z)(y\,\partial_{\mathrm{y}} - z\,\partial_{\mathrm{z}}), \text{ et}$$
$$Y = x\,\partial_{\mathrm{x}} + 2(1 + x^2 z)(y\,\partial_{\mathrm{y}} - z\,\partial_{\mathrm{z}}) - 2y^2 z\,\partial_{\mathrm{y}}$$

génèrent des feuilletages avec des holonomies conjuguées le long de l'axe x mais ne sont pas orbitalement analytiquement équivalents.

Notez que l'exemple ci-dessus correspond précisément au cas particulier (3.b) du corollaire B, avec $\mu = -2$ et $\lambda = 2$.

À la lumière de l'exemple ci-dessus, on rappelle le fait bien connu que l'holonomie calculée le long d'une *séparatrice faible* ne classe pas toujours un germe d'un feuilletage singulier. Le feuilletage du nœud-selle dans $(\mathbb{C}^2, 0)$ est probablement l'exemple le plus simple (voir, par exemple, [CCD13], Section 6.6.3). Cependant, à notre connaissance, l'exemple 1 est le premier illustrant que ce phénomène peut également se produire pour l'holonomie calculée le long d'une séparatrice forte.

Aperçu du texte

L'outil de base utilisé dans ce travail, le concept de séries $D_{r,R}$-transversalement formelles est présenté dans la section 2.1. Une *série $D_{r,R}$-transversalement formelle* est une série formelle de la forme

$$\sum_{k_i \in \mathbb{N}} f_K(x)\mathbf{z}^K, \tag{5.23}$$

où $\mathbf{z}^K = z^{k_1} \dots z^{k_n}$, et chaque coefficient $f_K(x)$ est convergent dans l'anneau $D_{r,R} := \{x \in \mathbb{C}; r < \|x\| < R\}$, où $r, R > 0$.

Dans la section 2.3, nous développons la théorie des endomorphismes dans l'anneau des séries $D_{r,R}$-transversalement formelles. En particulier, la motivation principale est de traiter les *dérivations $D_{r,R}$-transversalement formelles* (dérivations sur l'anneau des séries $D_{r,R}$-transversement formelles) comme des champs vectoriels avec des coefficients étant séries $D_{r,R}$-transversalement formelles.

Dans le chapitre 3, nous faisons une brève introduction à la théorie des champs vectoriels et des feuilletages. En particulier, nous formalisons de nombreuses notations qui seront utilisées par la suite. Une brève revue des résultats antérieurs sur la classification analytique et formelle des germes de champs vectoriels singuliers est faite dans la section 3.2. Dans la section 3.1, nous construisons l'holonomie d'une séparatrice donnée comme une application de la technique du relèvement de chemin. Dans la dernière section de ce chapitre, nous exposons les travaux des Elizarov et Il'ya-shenko, Helena Reis, et Mattei et Moussu sur les relations entre la conjugaison des holonomies et la classification analytique des feuilletages.

Au chapitre 4, nous étendons aux dérivations $D_{r,R}$-transversalement formelles, les notions d'application exponentielle, de forme normale et de symétries. Après cette étude générale, nous nous intéressons à la classe des *dérivations D_R-transversalement formelles* (dérivations sur l'anneau des séries de la forme (5.23), où chaque coefficient $f_K(x)$ est convergent dans le disque D_R) dans la section 4.3.

Enfin, au chapitre 5, nous prouvons le Théorème Principal et ses corollaires en dimension trois et quatre. Dans la section 5.5, nous étudions la nécessité de la propriété de non résonance transverse négative dans le Théorème Principal.

Résumé en français

Chapitre 1: Feuilletages réguliers et holonomie

Dans ce travail, nous étudions les feuilletages analytiques générés localement par des champs vectoriels. Plus précisément, les feuilles de tels feuilletages correspondent localement aux *courbes intégrales* des équations différentielles ordinaires associées à des champs vectoriels. Dans le premier chapitre, nous faisons une brève introduction à la théorie des champs vectoriels. En particulier, nous formalisons des nombreuses notations qui seront fréquemment utilisées.

Dans la deuxième section, nous faisons une construction en détail de l'*holonomie* d'un feuilletage régulier. La proposition principale dans ce chapitre affirme que le groupe d'holonomie d'une feuille compacte décrit le comportement transversal d'un feuilletage au voisinage de telle feuille. Nous pouvons énoncer ce résultat comme suit:

Proposition 1.2.13 ([CN85], Théorème 2, Chapitre IV). *Soient $\mathcal{L}_1$ et $\mathcal{L}_2$ des feuilles compactes C^s difféomorphes des feuilletages $\mathcal{F}_1$ et $\mathcal{F}_2$ respectivement. Les holo-nomies de $\mathcal{L}_1$ et $\mathcal{L}_2$ sont C^s conjuguées si et seulement si il existe des voisinages $V_1 \supset \mathcal{L}_1$ et $V_2 \supset \mathcal{L}_1$, et un difféomorphisme C^s $\Phi : V_1 \to V_2$, $\Phi(\mathcal{L}_1) = \mathcal{L}_2$, qui envoie des feuilles de $\mathcal{F}_1|_{V_1}$ sur des feuilles de $\mathcal{F}_2|_{V_2}$. Dans ce cas, on dit que $\mathcal{F}_1$ et $\mathcal{F}_2$ sont localement équivalents sur $\mathcal{L}_1$ et $\mathcal{L}_2$ et Φ est une équivalence locale.*

En particulier, l'objectif de ce travail est de généraliser ce résultat pour le cas singulier.

Chapitre 2: Endomorphismes, Automorphismes et dérivations

Dans ce chapitre, nous présentons un objet développé par Arame Diaw en 2019 [Dia19], les séries $D_{r,R}$-transversalement formelles. Précisément, l'anneau $\mathcal{O}_{r,R}[[\mathbf{z}]]$ des séries $D_{r,R}$-transversalement formelles est défini comme la limite inverse des espaces polynomiaux. Dans la Section 2.2, nous nous concentrons sur des sous-anneaux de $\mathcal{O}_{r,R}[[\mathbf{z}]]$ des séries $D_{r,R}$-transversalement convergentes $\mathcal{O}_{r,R}\{\mathbf{z}\}$ et D_R-transversalement convergentes $\mathcal{O}_R\{\mathbf{z}\}$.

Comme résultat le plus important de ce chapitre, nous établissons que l'anneau de la dernière ligne est exactement l'intersection des les deux dans la ligne médiane dans la Figure 5.9 ci-dessous. Dans la Section 2.3, nous développons la théorie des endomorphismes de $\mathcal{O}_{r,R}[[\mathbf{z}]]$. La motivation principale est de

$$\begin{array}{ccc} & \mathcal{O}_{r,R}[[\mathbf{z}]] & \\ \nearrow & & \nwarrow \\ \mathcal{O}_R[[\mathbf{z}]] & & \mathcal{O}_{r,R}\{\mathbf{z}\} \\ \nwarrow & & \nearrow \\ & \mathcal{O}_R\{\mathbf{z}\} & \end{array}$$

Figure 5.9: Schéma des inclusions

traiter des dérivations $D_{r,R}$-transversalement formelles (dérivations de $\mathcal{O}_{r,R}[[\mathbf{z}]]$) comme des champs vectoriels avec des coefficients dans $\mathcal{O}_{r,R}[[\mathbf{z}]]$ et des automorphismes $D_{r,R}$-transversalement formeles (automorphismes de $\mathcal{O}_{r,R}[[\mathbf{z}]]$) comme des changements des coordonnées avec des coefficients dans $\mathcal{O}_{r,R}[[\mathbf{z}]]$.

Chapitre 3: Feuilletages singuliers: holonomie et formes normales

La classification des feuilletages holomorphes singuliers est un sujet classique qui remonte aux travaux pionniers de Dulac et Poincaré (voir, par exemple, [Dul23]). La manière que Dulac et Poincaré avaient étudié cette classification est pour des formes normales des champs vectoriels. À un changement de coordonnées près, une forme normale d'un champ vectoriel est la manière la plus simple de le décrire. Cette théorie et également une petite partie du travail de Brjuno sont présentées dans la Section 3.1. Un des principaux résultats développée par Poincaré est le suivant :

Théorème 3.2.6. *Soit $X \in \mathcal{D}(\mathbb{C}[[\mathbf{z}]])$ une dérivation analytique. Si l'origine $0 \in \mathbb{C}$ n'appartient pas à $K(\lambda)$, alors X est analytiquement conjugué à sa forme normale.*

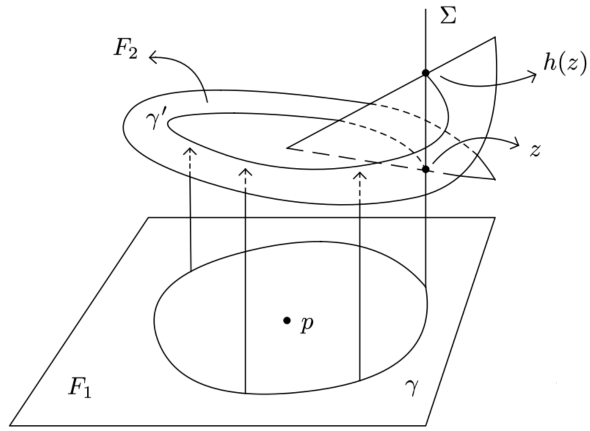
Dans la Section 3.3, nous montrons comment la technique du relèvement de chemin (voir Figure 5.10) peut être utilisée pour construire une équivalence entre des germes de feuilletages singuliers. Historiquement, cette technique est utilisé parallèlement avec la théorie de formes normales dans tous les résultats de classification des feuilletages déjà produits.

Pour illustrer l'usage de cette technique, nous avons donné l'idée de preuve du théorème suivant, démontré originalement par Mattei Moussu [MM80a].

Théorème 3.3.2. *Soit X un germe de champ vectoriel analytique singulier dans $0 \in \mathbb{C}^2$ avec des valeurs propres 1 et $\lambda \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Alors, X est analytiquement équivalent à sa partie linéaire si et seulement si l'holonomie d'une séparatrice du feuilletage $\mathcal{F}$ engendrée par X est linéarisable.*

Chapitre 4: Application exponentielle et crochet de Lie

La notion de dérivations $D_{r,R}$-transversalement formelles *nilpotentes* et leur exponentielle sont cruciales pour ce travail. Plus précisément, le *logarithme*,



La courbe γ' est le relèvement de γ sur la feuille F_2.

Figure 5.10: Le relèvement d'un chemin γ.

l'application inverse de l'application exponentielle, d'un changement de coordonnées *tangent à l'identité* est une dérivation nilpotente. Avec cette identification, nous pouvons prouver la formule suivante

$$(\exp X)^*Y = X + [X,Y] + \frac{1}{2}[X,[X,Y]] + \dots \tag{5.24}$$

Dans la première section, la notion de dérivation *nilpotente* de $\mathcal{D}(\mathcal{O}_{r,R}[[\mathbf{z}]])$, qui est le $\mathcal{O}_{r,R}[[\mathbf{z}]]$-module des dérivations de $\mathcal{O}_{r,R}[[\mathbf{z}]]$, est introduite. Dans la suite, nous explicitons la correspondance bijective entre les dérivations nilpotentes et les difféomorphismes qui sont donnés par *l'application exponentielle.*

Dans la section 4.2, de nombreux objets importants comme les applications ad et Ad et la *décomposition exponentielle* d'un changement de coordonnées $D_{r,R}$-transversalement formel *x-normalisée* sont définis. En particulier, ce dernier permet d'étendre, en quelque sorte, la Formule (5.24). Cependant, le concept de plus important introduit dans ce chapitre est celui de *symétrie* $D_{r,R}$-transversalement formelle. En effet, la technique développée par Arame Diaw [Dia19], qui est explorée ici, est basée sur les symétries des générateurs locaux des feuilletages.

Dans la section 4.3, nous calculons explicitement une forme normale pour des dérivations $D_{r,R}$-transversalement formelles. De plus, nous caractérisons le *centre* d'une dérivation D_R-transversalement formelle nilpotente x-normalisée qui n'ayant pas de résonance transverse négative. Plus précisément, la propriété de non résonance transversale négative énoncée dans l'introduction peut être reformulée comme suit.

Definition 4.3.4. *Nous dirons qu'une dérivation x-normalisée $X \in \mathcal{D}_{norm}(\mathcal{O}_R[[\mathbf{z}]])$*

n'a pas de résonance transversale négative si

$$\langle \mu, \mathcal{L}_n \rangle \cap \mathbb{Z}_{\geq 1} = \emptyset,$$

où $1, \mu_1, \ldots, \mu_n$ *sont les valeurs propres de* X *et* $\mu = (\mu_1, \ldots, \mu_n)$.

Chapitre 5: Caractérisation d'un feuilletage singulier par l'holonomie

Comme montré ci-dessus, si les holonomies de deux feuilletages réguliers sont analytiquement conjugués, alors il est possible de construire un changement de coordonnées qui envoie un feuilletage dans l'autre. En conséquence de ce résultat, nous montrons dans la section 5.2 qu'il est toujours possible de construire un *automorphisme* x*-normalisé* qui conjugue des générateurs locaux de feuilletages de type croisement $(\mathcal{F}, H, \Gamma)$ et $(\mathcal{G}, L, \Omega)$ sur un voisinage dans $D_{r,R}$ si les conditions suivantes sont satisfaites :

(a) Les parties linéaires des générateurs locaux de $(\mathcal{F}, H, \Gamma)$ et $(\mathcal{G}, L, \Omega)$ sont conjuguées.

(b) Le générateur local de $\mathcal{F}$ (et donc celui de $\mathcal{G}$) n'a pas de résonance transversalement négative.

(c) Les Γ-holonomie et Ω-holonomie sont analytiquement conjuguées.

À la fin de la section 5.2, nous montrons une hypothèse moins restrictive sur les valeurs propres des générateurs locaux que la condition *(a)*. La décision de faire *(a)* comme hypothèse principale n'était pas seulement pour garder la simplicité de l'énonce du Théorème Principal, mais aussi parce que cette condition ne représente aucune nouvelle restriction aux feuilletages équivalents. La section 5.3 contient la preuve du résultat principal de ce travail et son application en dimension 3.

Soient deux feuilletages de type croisement $(\mathcal{F}, H, \Gamma)$ et $(\mathcal{G}, L, \Omega)$. Si un automorphisme $\Phi : U \subset (\mathbb{C}^{n+1}, 0) \to V \subset (\mathbb{C}^{n+1}, 0)$ satisfait $\Phi(H \cap U) = L \cap V$ et $\Phi(\Gamma \cap U) = \Omega \cap V$, il est clair que, dans des coordenées adaptées, Φ préserve l'hypersurface lisse $\{x = 0\}$ et la courbe lisse $\{\mathbf{z} = 0\}$. Algébriquement, nous dirons qu'un automorphisme est $(x, \mathbf{z})$-*préservé* s'il satisfait satisfait

1. $\Phi(\langle x \rangle) \subset (\langle x \rangle)$,
2. $\Phi(\langle z_1, \ldots, z_n \rangle) \subset \langle z_1, \ldots, z_n \rangle$,

où $\langle x \rangle$ et $\langle z_1, \ldots, z_n \rangle$ sont respectivement l'ideal généré par les fonctions $f(x, \mathbf{z}) = x$ et $g_i(x, \mathbf{z}) = z_i$, pour $i \in \{1, \ldots, n\}$. Donc pour étudier l'équivalence analytique des feuilletages de type croisement, nous pouvons nous restreindre aux automorphismes $(x, \mathbf{z})$-préservés. Dans la Section 5.4, nous faisons l'étude de ce type d'automorphismes.

Dans la section 5.5, nous étudions la nécessité des conditions *(a)*, *(b)*, et *(c)*. En particulier, nous montrons que la condition *(b)* est presque optimale.

Plus précisément, soit une dérivation semi-simple $S = x\,\partial_{\mathrm{x}} + L(\mu)$ où $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$. Nous rappelons que associés à la dérivation S, on a les *ensembles résonnants négatifs et positifs* données par

$$\begin{aligned} \mathrm{NR}(S) &:= \{T \in \mathcal{L}_n \; ; \langle \mu, T\rangle \in \mathbb{Z}_{\geq 1}\}, \\ \mathrm{PR}(S) &:= \{K \in \mathcal{L}_n \setminus \{\mathbf{0}\} \; ; \langle \mu, K\rangle \in \mathbb{Z}_{\leq 0}\}. \end{aligned}$$

Nous démontrons que si un générateur local x-normalisé avec une partie semi-simple $S = x\,\partial_{\mathrm{x}} + L(\mu)$ a un ensemble non vide $\mathrm{NR}(S)$, alors, d'autres hypothèses, il y a deux champs vectoriels satisfaisant *(a)* et *(c)* ayant S comme partie semi-simple, mais qui ne génèrent pas des feuilletages analytiquement équivalents. Plus précisément, nous pouvons prouver le résultat suivant :

Proposition C. *Soit $\mu = (\mu_1, \ldots, \mu_n)$ un n-uplet dans $\mathbb{C}^n$ tel que $\mu_i \neq 1$ et $\mu_i \neq \mu_j$. Supposons qu'il existe un couple $(T, K) \in \mathrm{NR}(S) \times \mathrm{PR}(S)$ et un vecteur non nul $\lambda \in \mathbb{C}^n$ tel que*

1. $\langle \mu, T\rangle + \langle \mu, K\rangle \leq 0$,
2. *le champ vectoriel $\mathbf{z}^T L(\lambda)$ est analytique, et*
3. $\langle \lambda, K\rangle = 0$,

alors il existe deux champs vectoriels analytiques x-normalisés X et Y, avec S comme parties semi-simples, qui génèrent des feuilletages analytiques qui ont des holonomies conjuguées le long de l'axe x, mais qui ne sont pas analytiquement équivalents.

Comme conséquence de ce résultat, nous avons prouvé que la conjugaison des holonomies *n'implique pas* l'équivalence analytique des feuilletages comme dans le cas régulier.

Index

Bibliography

[Aba09] Marco Abate. *Local dynamics of singular holomorphic foliations.* ETS, 2009.

[BC09] James Ward Brown and Ruel V Churchill. *Complex variables and applications eighth edition.* McGraw-Hill Book Company, 2009.

[Bel14] Jordan Bell. The poincaré-dulac normal form theorem for formal vector fields. *USA: University of Toronto, Department of Mathematics*, 2014.

[CCD13] Felipe Cano, Dominique Cerveau, and Julie Déserti. *Théorie élémentaire des feuilletages holomorphes singuliers.* Belin, 2013.

[CN77] César Camacho and Alcides Lins Neto. Introdução à teoria das folheações. 1977.

[CN85] César Camacho and Alcides Lins Neto. *Geometric theory of foliations.* BIRKHAUSER, 1985.

[Dia19] Adjaratou Arame Diaw. *Géométrie de certains tissus holomorphes singuliers en dimension 2.* PhD thesis, Rennes 1, 2019.

[DL20] Adjaratou Arame Diaw and Frank Loray. Pairs of foliations and mattei-moussu's theorem. *arXiv preprint arXiv:2008.00719*, 2020.

[Dul12] Henri Dulac. Solutions d'un système d'équations différentielles dans le voisinage de valeurs singulières. *Bulletin de la Société mathématique de France*, 40:324–383, 1912.

[Dul23] Henri Dulac. Sur les cycles limites. *Bulletin de la Société Mathématique de France*, 51:45–188, 1923.

[EI84] PM Elizarov and Yu S Il'yashenko. Remarks on the orbital analytic classification of germs of vector fields. *Mathematics of the USSR-Sbornik*, 49(1):111, 1984.

[Hab18] H. Haber. Notes on the matrix exponential and logarithm. 2018.

[Hal00] Brian C Hall. An elementary introduction to groups and representations. *arXiv preprint math-ph/0005032*, 2000.

[Hum12] James E Humphreys. *Introduction to Lie algebras and representation theory*, volume 9. Springer Science & Business Media, 2012.

[IY08] Yulij Ilyashenko and Sergei Yakovenko. *Lectures on analytic differential equations*, volume 86. American Mathematical Soc., 2008.

[Lee13] John M Lee. Smooth manifolds. In *Introduction to Smooth Manifolds*, pages 1–31. Springer, 2013.

[Mar81a] Jean Martinet. Normalisation des champs de vecteurs holomorphes. In *Séminaire Bourbaki : vol. 1980/81, exposés 561-578*, number 23 in Séminaire Bourbaki, pages 55–70. Springer-Verlag, 1981. talk:564.

[Mar81b] Jean Martinet. Normalisation des champs de vecteurs holomorphes (d'après a.-d. brjuno [2]). In *Séminaire Bourbaki vol. 1980/81 Exposés 561–578*, pages 55–70. Springer, 1981.

[MM80a] J-F Mattei and Robert Moussu. Holonomie et intégrales premieres. In *Annales Scientifiques de l'École Normale Supérieure*, volume 13, pages 469–523, 1980.

[MM80b] Jean-François Mattei and Robert Moussu. Holonomie et intégrales premières. *Annales scientifiques de l'École Normale Supérieure*, 4e série, 13(4):469–523, 1980.

[MR82] Jean Martinet and Jean-Pierre Ramis. Problemes de modules pour des équations différentielles non linéaires du premier ordre. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 55(1):63–164, 1982.

[MR83] Jean Martinet and Jean-Pierre Ramis. Classification analytique des équations différentielles non linéaires résonnantes du premier ordre. In *Annales scientifiques de l'École normale supérieure*, volume 16, pages 571–621, 1983.

[Rei06] Helena Reis. Equivalence and semi-completude of foliations. *Nonlinear Analysis: Theory, Methods & Applications*, 64(8):1654–1665, 2006.

[RR11] Julio C Rebelo and Helena Reis. Local theory of holomorphic foliations and vector fields. *arXiv preprint arXiv:1101.4309*, 2011.

[Ser09] Jean-Pierre Serre. *Lie algebras and Lie groups: 1964 lectures given at Harvard University*. Springer, 2009.

[Sot11] Jorge Sotomayor. Equações diferenciais ordinárias. *Sao Paulo: Editora Livraria da Fısica*, 2011.

[Tan65] Shukichi Tanno. A theorem on regular vector fields and its applications to almost contact structures. *Tohoku Mathematical Journal, Second Series*, 17(3):235–238, 1965.

[Zhi72] Michail Zhitomirskii. *Smooth Local Normal Forms for Vector Fields and Diffeomorphisms*. Lecture notes IMPA, 1972.